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Homogenization of elliptic problems in thin domains with oscillatory boundaries

Homogenización de problemas elípticos en dominios finos con fronteras oscilantes

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Memoria para optar al título de doctor

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Dirigida por: José M. Arrieta Algarra

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A Jesús y Pilar, mis abuelos

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0.1. Introducción

Los dominios finos, es decir, dominios sustancialmente más pequeños en alguna o varias de sus direcciones que en el resto, aparecen en muchos campos de la ciencia. Por ejemplo, dinámica de fluídos (lubricación, conducción de fluídos en tubos delgados, dinámica de oceanos...), mecánica de sólidos (barras delgadas, placas o cáscaras) o incluso en fisiología (circulación de la sangre). Así, el amplio número de posibles aplicaciones a situaciones reales ha hecho que la investigación de modelos de ecuaciones en derivadas parciales en dominios finos se convierta en un tema muy estudiado en los últimos años.

Desde un punto de vista matemático, el estudio de las soluciones de una EDP en un dominio fino es un caso particular de la cuestión general relativa a cómo la variación de los dominios afecta al comportamiento de las soluciones de la EDP. En este marco, obtener la ecuación límite del modelo considerado, comparar la solución de la ecuación límite y las soluciones del problema en el dominio fino, analizar los coeficientes de la ecuación límite y comprender cómo la geometría del dominio afecta a la ecuación límite son algunos de los objetivos que deberían ser alcanzados. De hecho, es importante señalar que este tipo de cuestiones no sólo proporcionan importantes resultados teóricos sino que son muy relevantes desde el punto de vista de las aplicaciones. Por ejemplo, ser capaz de reducir el problema original a un problema mucho más sencillo, problema límite, que refleje las principales características del problema de partida puede ser muy útil para ingenieros y físicos.

En esta tesis, consideramos dominios finos en dos dimensiones con fronteras rugosas. Este tipo de dominios presenta la característica especial de que, además del grosor del dominio, otro pequeño parámetro comparado con el tamaño del dominio juega un papel muy importante en la descripción del problema límite: el orden del periodo de las oscilaciones. Así, para obtener la ecuación límite debemos tener esto muy en cuenta debido a la posible interacción entre las diferentes microescalas.

El estudio de problemas en dominios finos con fronteras oscilantes no es nuevo. En la literatura se pueden encontrar varios trabajos que tratan este tema. A continuación nos gustaría mencionar algunos de ellos y también referir al lector a sus correspondientes bibliografías. En [83, 84, 85] los autores estudian el comportamiento de las soluciones de ciertos problemas elípticos y parabólicos en dominios finos perforados con frontera oscilante. Los resultados obtenidos en estos artículos están relacionados con la construcción de una adecuada expansión asintótica de las soluciones. Este enfoque para abordar el problema fue propuesto por T. Melnyk en [82] para la investigación de problemas elípticos y espectrales en dominios finos perforados.

En el contexto de la mecánica de fluídos nos gustaría citar [38] donde se estudia el comportamiento de las soluciones del sistema de Navier-Stokes en un domino fino con frontera rugosa. Otros trabajos interesantes donde se analiza el efecto de fronteras rugosas en el comportamiento de flujos de fluidos son [4, 29, 66, 76, 77].

Las estructuras finas con fronteras rugosas han sido muy estudiadas desde el punto de vista de la elasticidad. Por ejemplo, en [25, 26] se describe el comportamiento de una estructura elástica compuesta por un conjunto de barras elásticas y una placa delgada. También referimos al lector a [5, 16, 28] donde se han estudiado películas finas con perfiles muy oscilantes.

Otros trabajos interesantes tratando con ecuaciones en dominios finos con fronteras oscilantes son [18, 41, 42].

El propósito de esta tesis es analizar el comportamiento de las soluciones de ciertas ecuaciones elípticas definidas en dominios finos con fronteras rugosas más allá de las hipótesis clásicas de periodicidad. Para ser más precisos, consideraremos dominios finos con fronteras localmente periódicas y dominios finos donde la frontera superior e inferior oscilan pero no necesariamente con la misma frecuencia y perfil de oscilación. En particular, estamos interesados en entender cómo la geometría a nivel microscópico del dominio afecta al comportamiento de las soluciones. Así, obtendremos el problema límite para una ecuación modelo en diferentes dominios finos con fronteras oscilantes y estudiaremos los efectos de la estructura microscópica sobre el comportamiento macroscópico de las soluciones.

Éste es precisamente el principal objetivo de la teoría matemática de la homogeneización. La homogeneización estudia el comportamiento macróscopico de un modelo (típicamente una ecuación en derivadas parciales o un funcional de energía) que es "microscópicamente" heterogéneo para describir algunas características del medio heterogéneo. Por ejemplo, la teoría de la homogeneización se encarga del estudio de ecuaciones en derivadas parciales con coeficientes oscilantes dependientes de una escala pequeña o de ecuaciones en derivadas parciales definidas en dominios con geometrías que dependen fuertemente de un parámetro muy pequeño (dominios perforados, dominios con fronteras oscilantes, etc.). El origen de la palabra homogeneización está relacionado con la idea de reemplazar un material heterogéneo por uno "equivalente" homogéneo. Esto quiere decir desde un punto de vista matemático, y hablando de una forma muy general, que las soluciones de un problema de valor en la frontera dependiendo de un parámetro pequeño convergen en cierto sentido a la solución de un problema límite que es descrito explícitamente.

Podemos distinguir cuatro ramas dentro de la teoría de la homogeneización: G o H-convergencia [90, 91, 105, 107, 104], la segunda está relacionada con descripciones probabilísticas y estocásticas de medios heterogéneos [20, 94], la Γ-convergencia [63, 62, 58], y la cuarta surge para estudiar estructuras periódicas [19, 101, 52].

Diferentes métodos han sido desarrollados en el marco de la homogeneización periódica. Por ejemplo, métodos de expansión asintótica junto con el método de las funciones oscilantes de Tartar [19, 101, 44], el método de la convergencia en dos escalas [2, 92], que puede ser interpretado como una convergencia intermedia entre la convergencia débil y fuerte en L^p con la capacidad de capturar oscilaciones rápidas en una escala muy pequeña. Más reciente es el método conocido por "unfolding periodic method" introducido por D. Cioranescu, A. Damlamian and G. Griso en [45, 46] del que hablaremos más en detalle después. El "unfolding periodic method" está muy relacionado con la técnica de dilatación [6, 34].

Nótese que los métodos surgidos para tratar problemas en estructuras periódicas son muy eficaces aunque menos generales que otros. Después de todo, la hipótesis de periodicidad es muy restrictiva. Sin embargo, estos métodos han sido muy utilizados a lo largo de los años ya que presentan importantes ventajas. Primero de todo, las hipótesis de periodicidad son el marco más sencillo posible con el que trabajar. Por ejemplo, el método de la expansión asintótica en dos escalas nos permite obtener el problema límite e incluso estimaciones de error de una manera muy sencilla bajo las hipótesis de periodicidad. Segundo, nos permiten analizar modelos muy complicados que no son fácilmente tratables con métodos más generales. Además, como veremos en esta tesis, su importancia va más allá de las hipótesis clásicas de periodicidad. En particular, veremos como adaptando métodos surgidos para estructuras periódicas podemos analizar casos más generales: fronteras localmente periódicas, la amplitud y el periodo de las oscilaciones varían en espacio, y dominios finos con fronteras oscilantes con diferentes perfiles y frecuencia de oscilación.

Antes de explicar el objetivo y los contenidos de la tesis nos gustaría mencionar algunos artículos que han motivado esta investigación. Entonces, para contextualizar nuestro trabajo vamos a describir los principales resultados obtenidos en [7, 8, 9, 10]. Para hacer esto, analizaremos brevemente el comportamiento de las soluciones del siguiente problema modelo

$$\begin{cases} -\Delta w^\epsilon + w^\epsilon = f & \text{in } R^\epsilon, \\ \dfrac{\partial w^\epsilon}{\partial \nu^\epsilon} = 0 & \text{on } \partial R^\epsilon, \end{cases} \tag{0.1.1}$$

donde $f \in L^2(0,1)$, ν^ϵ es la normal unitaria exterior ∂R^ϵ y R^ϵ es un dominio fino en dos dimensiones con grosor de orden ϵ y con una frontera oscilante que es definido como sigue

$$R^\epsilon = \Big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ 0 < y < \epsilon\, G_\epsilon(x)\Big\}, \tag{0.1.2}$$

siendo $G_\epsilon(\cdot)$ una función oscilante dependiendo del parámetro ϵ. Para simplificar la presentación de los resultados asumimos que $G_\epsilon(\cdot)$ es del siguiente tipo

$$G_\epsilon(x) = a(x) + g(x/\epsilon^\alpha) \tag{0.1.3}$$

donde $\alpha > 0$ es un parámetro, $a : \mathbb{R} \to \mathbb{R}$ es una función positiva y regular y $g : \mathbb{R} \to \mathbb{R}$ es una función regular $L-$periódica que verifica $0 < g_0 \leq g(\cdot) \leq g_1$ para ciertas constantes positivas g_0 y g_1.

Obsérvese que la existencia y unicidad de las soluciones del problema (0.1.1) están garantizadas por el Teorema de Lax-Milgram para cada $\epsilon > 0$.

Como veremos a continuación, el comportamiento de las soluciones dependerá esencialmente del valor del parámetro α. Además, teniendo en cuenta que el grosor del dominio R^ϵ tiende a cero, cabe esperar que la familia de soluciones w^ϵ converja a una función en una sola variable cuando el parámetro ϵ tiende cero, es decir, la función límite no dependerá de la variable y.

Antes de nada, nos gustaría señalar que si el dominio fino no presenta oscilaciones $G_\epsilon(x) = G(x)$, $g \equiv 0$ o $\alpha \equiv 0$ en (0.1.3), se sabe que la ecuación límite es

$$\begin{cases} -\dfrac{1}{G(x)}\big(G(x)w_x(x)\big)_x + w(x) = f, \quad x \in (0,1), \\ w_x(0) = w_x(1) = 0. \end{cases}$$

Este resultado puede ser visto en [73, 100]. La manera estándar de obtener el problema límite en este caso es realizar un cambio de variable, $x_1 = x$, $x_2 = \epsilon\, G(x)y$, que transforme el dominio fino en un dominio independiente de ϵ, $R = (0,1)\times(0,1)$. Entonces, el parámetro ϵ aparece en los coeficientes del nuevo operador diferencial y no es difícil pasar al límite usando la formulación débil del nuevo problema.

Las técnicas presentadas en [7] para estudiar el comportamiento de las autofunciones y los autovalores en dominios finos con fronteras oscilantes pueden ser fácilmente adaptadas para obtener el problema límite de (0.1.1) cuando el dominio fino presenta oscilaciones débiles, es decir, $0 < \alpha < 1$ en (0.1.3). Entonces, teniendo en cuenta las siguientes convergencias

$$G_\epsilon(x) = a(x) + g\Big(\frac{x}{\epsilon^\alpha}\Big) \overset{\epsilon\to 0}{\rightharpoonup} m(x) := a(x) + \frac{1}{L}\int_0^L g(s)\,ds \quad w - L^2(0,1),$$
$$\frac{1}{G_\epsilon(x)} = \frac{1}{a(x) + g\Big(\frac{x}{\epsilon^\alpha}\Big)} \overset{\epsilon\to 0}{\rightharpoonup} k(x) := \frac{1}{L}\int_0^L \frac{1}{a(x)+g(s)}\,ds \quad w - L^2(0,1),$$

se obtiene que el problema límite es

$$\begin{cases} -\dfrac{1}{m(x)}\Big(\dfrac{1}{k(x)}w_x\Big)_x + w = f, \quad x \in (0,1), \\ w_x(0) = w_x(1) = 0. \end{cases}$$

Referimos al lector al Capítulo 4 de [7] para los detalles y algunas generalizaciones. La prueba de este resultado consiste en reducir el problema en dos dimensiones a uno en una dimensión con el coeficiente de difusión oscilante pero en el que no es difícil pasar al límite. Cabe señalar que el ingrediente clave para que esta técnica funcione es que

$$\lim_{\epsilon\to 0} \epsilon\, G'_\epsilon(x) \overset{\epsilon\to 0}{\longrightarrow} 0, \text{ uniformemente en } (0,1),$$

que en este ejemplo se verifica ya que $0 < \alpha < 1$.

Los autores en [8, 9] estudian el caso $\alpha = 1$ en (0.1.3). Es interesante destacar que la altura del dominio, la amplitud y el periodo de las oscilaciones son del mismo orden en este caso, orden ϵ. Esto hace el problema muy resonante y que la determinación de la ecuación límite no sea sencilla.

El caso puramente periódico, $G_\epsilon(x) = g(x/\epsilon)$ en (0.1.3), fue estudiado en [8], también en [84]. Los autores analizan no sólo la ecuación elíptica sino también el correspondiente problema semilineal parabólico. Cabe destacar que este problema está dentro de las hipótesis clásicas de la teoría de homogeneización periódica ya que la frontera oscilante está defina por una función $L-$periódica y, en consecuencia, existe una celda que describe la geometría del dominio

$$Y^* = \{(y_1, y_2) \in \mathbb{R}^2 \ : \ 0 < y_1 < L, \ 0 < y_2 < g(y_1)\}.$$

Entonces, en [8] el problema límite fue obtenido usando técnicas clásicas de homogeneización, especialmente las relacionadas con las estructuras reticuladas, referimos al lector a [19, 101, 44, 104] para una presentación de la teoría clásica de homogeneización y a [52] para ver resultados más enfocados a las estructuras reticuladas. En particular, el problema límite fue obtenido formalmente usando el método de múltiples escalas y, posteriormente, los autores usaron la técnica de las funciones oscilantes de Tartar para probar el teorema de convergencia. Es importante señalar que, al igual que en el caso de dominios perforados, para obtener el resultado de homogeneización es necesario construir un operador de extensión ya que la geometría del dominio depende de ϵ, y además, no hay relaciones de inclusión entre los espacios $L^2(R^\epsilon)$ y $L^2(R^{\epsilon'})$ si $\epsilon \neq \epsilon'$.

Entonces, el problema límite de (0.1.1) en este caso particular es

$$\begin{cases} -q_0 w_{xx} + w = f(x), \quad x \in (0,1), \\ w'(0) = w'(1) = 0 \end{cases}$$

donde

$$q_0 = \frac{1}{|Y^*|} \int_{Y^*} \Big\{1 - \frac{\partial X}{\partial y_1}(y_1, y_2)\Big\} dy_1 dy_2,$$

y X es la única solución $L-$periódica en la primera variable del siguiente problema

$$\begin{cases} -\Delta X = 0 \text{ in } Y^*, \\ \dfrac{\partial X}{\partial N} = 0 \text{ on } B_2, \\ \dfrac{\partial X}{\partial N} = -\dfrac{g'(y_1)}{\sqrt{1 + g'(y_1)^2}} \text{ on } B_1, \\ \displaystyle\int_{Y^*} X(y_1, y_2)\, dy_1 dy_2 = 0, \end{cases}$$

donde B_1 es la frontera superior y B_2 es la frontera inferior Y^*.

Obsérvese que la geometría del dominio entra en la ecuación límite a través del coeficiente de difusión. De hecho, el coeficiente de difusión depende de la función X que a su vez es una función armónica definida en la celda representativa Y^* que describe la geometría del dominio fino.

Una generalización del caso anterior fue estudiada en [9]. En este artículo los autores incluyen el caso en el que la amplitud de las oscilaciones depende de la variable x de una forma regular. Por ejemplo, para fijar ideas podemos pensar que la función que define la frontera oscilante en el dominio fino (0.1.2) es definida como en (0.1.3) con $\alpha = 1$.

El problema límite es

$$\begin{cases} -\dfrac{1}{p(x)}(q(x)w_x)_x + w = f(x), \quad x \in (0,1), \\ w'(0) = w'(1) = 0, \end{cases}$$

donde

$$q(x) = \int_{Y^*(x)} \Big\{1 - \frac{\partial X(x)}{\partial y_1}(y_1, y_2)\Big\} dy_1 dy_2,$$
$$p(x) = |Y^*(x)|,$$

y $X(x)$ es la única solución $L-$periódica en la primera variable del siguiente problema

$$\begin{cases} -\Delta X(x) = 0 \text{ in } Y^*(x), \\ \dfrac{\partial X(x)}{\partial N} = 0 \text{ on } B_2(x), \\ \dfrac{\partial X(x)}{\partial N} = N_1(x) \text{ on } B_1(x), \\ \displaystyle\int_{Y^*(x)} X(x)\, dy_1 dy_2 = 0, \end{cases}$$

con $B_1(x)$ siendo la frontera superior y $B_2(x)$ la frontera inferior de la celda $Y^*(x)$ dada por

$$Y^*(x) = \{(y_1, y_2) \in \mathbb{R}^2 \ : \ 0 < y_1 < L, \quad 0 < y_2 < a(x) + g(y_1)\}, \ \forall x \in (0,1).$$

Se observa que la dependencia en la variable x de la amplitud de las oscilaciones se manifiesta explícitamente en el problema límite. De hecho, el coeficiente de la ecuación límite depende de una familia de problemas auxiliares definidos en una familia uniparamétrica de celdas que capturan de alguna manera la geometría localmente periódica del dominio.

Vale la pena señalar que ninguna de las técnicas usadas en los casos anteriores pueden ser aplicadas a esta situación debido al carácter localmente periódico del dominio. Por tanto, en [9] los autores proponen un método muy interesante que combina técnicas de la teoría de homogeneización con resultados de perturbación de dominios. En efecto, primero resuelven el caso periódico a trozos y después mediante un argumento de aproximación obtienen la ecuación límite para el caso general. En este trabajo se prueba mediante técnicas de perturbación de dominios que las soluciones de (0.1.1) dependen continuamente de las funciones G_ϵ las cuales definen la frontera oscilante.

Además, la cuestión de correctores para los dos problemas explicados anteriormente, $\alpha = 1$ en (0.1.3) fue analizada en [96, 97]. Referimos al lector a [19, 44, 93] para ver una introducción sobre correctores y estimaciones de error en problemas de homogeneización periódica.

Finalmente, en el artículo [10] se estudia el caso donde el dominio fino presenta oscilaciones muy rápidas, es decir, $\alpha > 1$ en (0.1.3). Aquí la rugosidad de la frontera es tan extrema que se considera la siguiente división natural del dominio

$$\begin{aligned} R_+^\epsilon &= \Big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ \epsilon\, G_0(x) < y < \epsilon\, a(x) + \epsilon\, g(x/\epsilon^\alpha)\Big\}, \\ R_-^\epsilon &= \Big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ 0 < y < \epsilon\, G_0(x)\Big\}. \end{aligned}$$

donde para cada $x \in (0,1)$ la función G_0 es definida como sigue

$$G_0(x) = \min_{y_1 \in \mathbb{R}} \{a(x) + g(y_1)\} = a(x) + g_0.$$

Entonces, para obtener el problema límite los autores analizan las propiedades de las soluciones en cada una de las partes y, teniendo en cuenta las características específicas de las soluciones en cada parte, construyen funciones test adecuadas para pasar al límite. En este caso le ecuación límite es

$$\begin{cases} -\dfrac{L}{\int_0^L \big(a(x) + g(y)\big)\, dy} \Big(\big(a(x) + g_0\big) w_x\Big)_x + w = f(x), & x \in (0,1) \\ w'(0) = w'(1) = 0. \end{cases}$$

Por tanto, queda claro que los artículos anteriormente citados han contribuído a resolver importantes problemas relacionados con el estudio del comportamiento de las soluciones de una ecuación en derivadas parciales definida en un dominio fino con frontera oscilante. Sin embargo, interesantes cuestiones surgen en este punto:

- ¿Es posible obtener los mismos problemas límites asumiendo menos regularidad en la frontera oscilante de los dominios finos?
- ¿Cuáles son los correctores que nos permiten obtener convergencias fuertes en el caso de dominios finos con fronteras que presentan oscilaciones débiles y en el caso de fronteras con oscilaciones rápidas?
- En el contexto de dominios finos localmente periódicos, ¿cúal es la influencia de un periodo variable en el comportamiento macroscópico de las soluciones?
- ¿Cúal es el problema límite homogeneizado para dominios finos que presentan oscilaciones en la frontera superior e inferior?
- ¿Las pruebas de los resultados de homogeneización para dominios con fronteras doblemente oscilantes son simples adaptaciones de las técnicas conocidas o presentan alguna dificultad extra?

Así, la motivación de esta tesis ha sido responder a estas cuestiones e incrementar el conocimiento de la teoría elíptica lineal en problemas definidos en dominios finos con fronteras oscilantes.

0.2. Objetivos

El principal objetivo de esta tesis es estudiar el comportamiento de las soluciones de ciertos problemas elípticos definidos en dominios finos con fronteras oscilantes que van mas allá de las clásicas hipótesis de periodicidad. En concreto, estudiaremos el caso de dominios finos con fronteras oscilantes localmente periódicas donde tanto la amplitud como el periodo de las oscilaciones varían en espacio; también analizaremos algunos casos de dominios finos donde la frontera superior e inferior oscilan con diferentes perfiles y frecuencia de oscilación.

0.3. Contenidos

En el Capítulo 1 adaptamos el método conocido como "unfolding operator method", véase [45, 59, 46], al estudio de dominios finos con fronteras periódicas oscilantes. El método "unfolding" ha sido aplicado con éxito a muchos problemas. Por ejemplo, en [48] esta técnica se usó para para estudiar la homogeneización de dominios con agujeros, en [71, 72] se obtienen estimaciones de error para problemas de homogeneización periódica, también se utilizó para estudiar problemas definidos en dominios con fronteras rugosas, véase [60, 24, 25, 26, 38] y para otras interesantes aplicaciones referimos al lector a [70, 47, 65, 11, 49]. Una de las principales ideas del método "unfolding" es introducir el llamado operador "unfolding" (parecido al operador de dilatación, véase [6, 34, 35, 36]), basado en un cambio de escala que permite obtener resultados de homogeneización con hipótesis muy poco restrictivas sobre la regularidad de la estructura.

El propósito principal de este capítulo es introducir el método del "unfolding operator" dando un enfoque general que nos permita analizar de una manera unificada los diferentes casos puramente periódicos. Así, en este capítulo analizaremos el comportamiento de las soluciones de la ecuación de Poisson con condiciones de Neumann homogéneas en dominios finos definidos como

$$R^\epsilon = \Big\{(x, y) \in \mathbb{R}^2 \mid x \in (0, 1),\ 0 < y < \epsilon\, g(x/\epsilon^\alpha)\Big\},$$

donde $\alpha > 0$ y $g : \mathbb{R} \to \mathbb{R}$ es una función $L-$periódica verificando que $0 < g_0 \leq g(\cdot) \leq g_1$ para ciertas constantes g_0 y g_1. En particular, en la Sección 1.1 introducimos el operador "unfolding" para este tipo de dominios finos y mostramos sus principales propiedades mientras que en las siguientes tres secciones obtenemos el problema límite y nuevos resultados de correctores dependiendo del orden de frecuencia de las oscilaciones.

Es importante tener en cuenta que el método desarrollado en este capítulo no sólo nos proporciona una técnica fácilmente aplicable para la homogeneización de ecuaciones en derivadas parciales definidas en dominios finos, sino que también nos permite reemplazar la regularidad sobre la frontera oscilante requerida en trabajos previos por la existencia de una desigualdad de Poincaré$-$Wirtinger en la celda representativa. Por tanto, como mostramos en la introducción del Capítulo 1, podremos considerar dominios finos más generales incluso con fronteras no continuas.

Además, creemos que este primer capítulo sirve como introducción para conseguir una mejor comprensión del capítulo siguiente donde el método será extendido a casos localmente periódicos.

El Capítulo 2 trata con el mismo tipo de ecuaciones elípticas que el capítulo anterior pero ahora la frontera oscilante del dominio en dos dimensiones es localmente periódica. En efecto, alejándonos del caso periódico la amplitud y periodo de las oscilaciones pueden no ser constantes y, por tanto, variar en espacio. Como ejemplo, a lo largo del capítulo consideraremos casos más generales, podemos asumir que el dominio fino es dado por

$$R^\epsilon = \big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\, 0 < y < \epsilon\,\big(a(x) + g(\frac{x}{l(x)\epsilon})\big)\big\}, \tag{0.3.1}$$

donde $a : (0,1) \to \mathbb{R}$ y $l : \mathbb{R} \to \mathbb{R}$ son funciones regulares positivas y acotadas y $g : \mathbb{R} \to \mathbb{R}$ es una función regular 1-periódica.

Nótese que, rigurosamente hablando, no existe una celda representativa para este dominio ya que las propiedades de periodicidad varían de un punto a otro. Sin embargo, por analogía con el caso periódico, nos referiremos al dominio definido a continuación como celda básica

$$Y^*(x) = \{(y_1, y_2) \in \mathbb{R}^2 \;:\; 0 < y_1 < l(x), \quad 0 < y_2 < a(x) + g\big(y_1/(l(x)\big)\}, \tag{0.3.2}$$

Obsérvese también que las hipótesis asumidas en este capítulo generalizan el caso analizado en [9], no sólo se incluyen fronteras localmente periódicas con periodo constante sino que también se permite que el periodo dependa de la variable x. Además, esta nueva situación con el periodo variable no es una simple generalización del caso presentado en [9]. Por un lado, no está claro cómo adaptar el método introducido en [9] al caso con periodo variable, ya que los argumentos expuestos usan de una manera esencial que el periodo es constante. Por otro lado, no es posible determinar de qué forma afecta el periodo variable al problema límite a priori, incluso conociendo los resultados de [9]. En un principio no es obvio si el problema límite correspondiente a (0.1.1) con R^ϵ dado por (0.3.1) debería ser

$$-\frac{1}{|Y^*(x)|}\big(r(x)w_x\big)_x + w = f, \qquad \text{o} \qquad -\frac{l(x)}{|Y^*(x)|}\Big(r(x)\Big(\frac{1}{l(x)}w\Big)_x\Big)_x + w = f,$$

donde $r(x) = \displaystyle\int_{Y^*(x)} \Big\{1 - \frac{\partial X(x)}{\partial y_1}(y_1, y_2)\Big\} dy_1 dy_2$, o quizás

$$-\frac{l(x)}{|Y^*(x)|}\big(r(x)w_x\big)_x + w = f,$$

con $r(x) = \displaystyle\frac{1}{l(x)}\int_{Y^*(x)} \Big\{1 - \frac{\partial X(x)}{\partial y_1}(y_1, y_2)\Big\} dy_1 dy_2$, o incluso otro. Nótese que todas estas ecuaciones coinciden si se considera el periodo constante.

Por tanto, el propósito de este capítulo es dar herramientas que permitan analizar la influencia de estructuras localmente periódicas a nivel microscópico en las propiedades macroscópicas del sistema.

Es importante destacar la relevancia que tiene el estudio de este tipo de problemas desde el punto de vista de las aplicaciones. Como ya hemos dicho al principio de esta introducción, las estructuras finas aparecen en muchos campos científicos y sirven para modelizar muchos fenómenos del mundo real. Además, este tipo de estructuras delgadas pueden ser vistas como casos particulares de un campo de estudio más amplio que también considera estructuras reticuladas o dominios perforados y que es muy relevante para el análisis de materiales compuestos. Entonces, comprender los efectos de la microestructura en las propiedades macro del material se convierte en un auténtico desafío. En una primera aproximación se puede suponer que el material a nivel microscópico es completamente periódico. Sin embargo, en muchas situaciones reales esto no es del todo realista y las estructuras presentan propiedades de periodicidad en la escala micro que varían en la escala macro y, por tanto, son distintas de un punto a otro, referimos al lector a [31, 53, 99, 103] para ver diferentes ejemplos de problemas reales en estructuras que no son puramente periódicas. Por tanto, es muy interesante desde el punto de vista de las aplicaciones dar herramientas que contribuyan a analizar la influencia de las microestructuras localmente periódicas en las propiedades macro.

De hecho, aunque la cantidad de trabajos de investigación en el campo de la homogeneización es bastante grande, sólo unos pocos se enmarcan dentro de la homogeneización de estructuras localmente periódicas. Entre otros, podemos citar [39, 88, 89] donde se usa una técnica de expansion asintótica para obtener el problema límite y estimaciones de la tasa de convergencia para problemas definidos en dominios localmente perforados. La convergencia en dos escalas se aplicó en [40, 79] para estudiar diferentes problemas en dominios con agujeros variando en espacio. En [99] la convergencia en dos escalas fue generalizada a medios fibrados localmente periódicos. Otros trabajos interesantes son [31, 32, 1, 80].

A pesar de los trabajos mencionados en el párrafo anterior, tenemos que decir que el caso introducido en este capítulo no ha sido tratado en la literatura. Para conseguir nuestro objetivo y obtener el problema límite para la situación general “localmente periódica” donde no sólo la amplitud sino también el periodo dependen de la variable x hemos extendido el método del “unfolding operator”. Vale la pena destacar que la construcción del nuevo operador “unfolding” no es trivial y sus propiedades no son una simple adaptación de los equivalentes resultados para el caso periódico. Referimos al lector a las secciones 2.2 y 2.3 para ver en detalle las importantes diferencias que existen respecto del operador definido en el capítulo anterior para modelos periódicos.

Entonces, en las Secciones 2.4 y 2.5 obtenemos el problema límite y un resultado de correctores usando las propiedades del operador “unfolding” probadas en las secciones anteriores. Si el dominio es dado por (0.3.1) se prueba que la ecuación límite asociada al problema (0.1.1) es

$$\begin{cases} -\dfrac{l(x)}{|Y^*(x)|}\big(r(x)w_x\big)_x + w = f, & x \in (0,1), \\ w'(0) = w'(1) = 0, \end{cases} \tag{0.3.3}$$

donde
$$r(x) = \frac{1}{l(x)} \int_{Y^*(x)} \Big\{1 - \frac{\partial X(x)}{\partial y_1}(y_1, y_2)\Big\} dy_1 dy_2,$$
y $X(x,\cdot,\cdot)$ es la única solución que es $l(x)-$periódica en la primera variable del correspondiente problema auxiliar en la celda $Y^*(x)$ dada por (0.3.2).

Es importante resaltar que esta extensión a ciertas estructuras localmente periódicas deja clara la versatilidad y el potencial del "unfolding method" para resolver problemas de homogeneización. Además creemos que la extensión del método presentada en este capítulo puede arrojar luz sobre cómo tratar otros problemas relacionados en diferentes estructuras.

El Capítulo 3 se dedica al estudio de las soluciones de la ecuación de Poisson con condiciones de Neumann homogéneas definas en dominos finos con frontera doblemente oscilante. En realidad, el motivo de estudiar este tipo de dominios es que se trata de la extensión natural de los modelos estudiados en el Capítulo 1 a una situación más realista donde pueden aparecer varias escalas microscópicas.

En particular, en este capítulo se consideran dominios finos que presentan oscilaciones de amplitud ϵ en la frontera superior e inferior. Para fijar ideas podemos considerar en esta Introducción el siguiente modelo de dominio fino

$$R^\epsilon = \Big\{(x, y) \in \mathbb{R}^2 \mid x \in (0, 1),\ -\epsilon\, h(x/\epsilon^\alpha) < y < \epsilon\, g(x/\epsilon^\beta)\Big\}, \quad \text{con } \beta, \alpha > 0. \tag{0.3.4}$$

donde $g, h : \mathbb{R} \to \mathbb{R}$ son C^1 funciones periódicas de periodo L_1 y L_2 repectivamente. Además, existen constantes $h_0 \geq 0$ y $h_1, g_0, g_1 > 0$ tal que $0 \leq h_0 \leq h(\cdot) \leq h_1$, y $0 < g_0 \leq g(\cdot) \leq g_1$.

En este contexto estamos interesados en analizar minuciosamente cómo el problema límite captura el diferente comportamiento oscilatorio de las dos fronteras. Nótese que, como principal novedad, permitimos que la frontera superior e inferior presenten diferentes perfiles y frecuencia de oscilación. Por tanto, podemos distinguir seis diferentes comportamientos dependiendo del valor de los parámetros α y β: oscilaciones rápidas y resonantes ($\alpha > 1, \beta = 1$), oscilaciones rápidas y débiles ($\alpha > 1, \beta < 1$), oscilaciones rápidas en la frontera superior e inferior($\alpha, \beta > 1$), oscilaciones débiles en la frontera superior e inferior ($\alpha, \beta < 1$), oscilaciones débiles y resonantes (($\alpha < 1, \beta = 1$)) y el caso resonante ($\alpha = \beta = 1$). En este capítulo estudiaremos los cuatro primeros casos. En estos momentos estamos trabajando en los dos casos restantes para completar el estudio.

Es importante señalar que las técnicas introducidas en este capítulo no son una simple generalización de las técnicas aplicadas a dominios finos con solo una frontera oscilante. De hecho, en este capítulo mostramos cómo combinar adecuadamente diferentes técnicas para obtener el problema límite y resultados de correctores para los diferentes casos considerados.

Siendo más precisos, en la Sección 3.1 analizamos el caso de dominios finos con fronteras rápidas y resonantes combinando métodos clásicos de homogeneización periódica y resultados de perturbaciones. En particular, obtenemos el problema límite y establecemos importantes propiedades de convergencia para las soluciones. Así, la ecuación límite correspondiente al problema 0.1.1 donde el dominio fino es definido en (0.3.4) con $\beta = 1$ y $\alpha > 1$ es dado por

$$\begin{cases} -\dfrac{\hat{q}}{\frac{|Y^*|}{L_1} + p} w_{xx} + w = f(x), \quad x \in (0,1) \\ w'(0) = w'(1) = 0 \end{cases}$$

donde

$$\hat{q} = \frac{1}{L_1} \int_{Y^*} \Big\{ 1 - \frac{\partial X}{\partial y_1}(y_1, y_2) \Big\} dy_1 dy_2, \quad p = \frac{1}{L_2} \int_0^{L_2} h(s) ds - h_0,$$

y X es una función armónica apropiadamente definida en la celda representativa Y^* correspondiente a las oscilaciones de la frontera superior.

Obsérvese que el problema límite refleja el comportamiento oscilatorio de la frontera del dominio fino. De hecho en el coeficiente de difusión se distinguen claramente dos tipos de términos: $\hat{q}$ relacionado con las oscilaciones resonantes y p con las oscilaciones rápidas.

Referimos al lector a la Subsección 3.1.1 y la Subsección 3.1.2 para una descripción detallada de los resultados. Además, la cuestión de correctores se trata en la Subsección 3.1.3. Cabe destacar que la construcción de la función corrector no es estándar y nos permite deducir información importante de cómo la frontera oscilante afecta al comportamiento de las soluciones.

Para concluir esta sección generalizamos los resultados obtenidos a ciertos dominios finos perforados con frontera doblemente oscilante, ver Subsección 3.1.4.

Las siguientes dos secciones completan el estudio de dominios finos con al menos una frontera con oscilaciones rápidas. En concreto, combinamos el método del "unfolding operator" introducido en el Capítulo 1 con la apropiada elección de funciones test oscilantes para obtener el problema límite correspondiente a los dos casos siguientes: dominios finos con oscilaciones rápidas en la frontera inferior y débiles en la frontera superior, $\alpha > 1$ y $0 < \beta < 1$ in (0.3.4), y dominios finos con oscilaciones rápidas en las dos fronteras $\alpha, \beta > 1$ en (0.3.4).

En las Sección 3.2 se estudia el caso con oscilaciones rápidas y débiles, $\alpha > 1$ y $\beta < 1$, y en la Sección 3.3 el caso con oscilaciones rápidas en las dos fronteras. Por tanto, se prueba que el problema límite correspondiente a (0.1.1) con el modelo de dominio fino definido en (0.3.4) es dado por

- Oscilaciones rápidas y débiles, $(0 < \beta < 1)$.

$$\begin{cases} -\dfrac{1}{\mathcal{M}\big(\frac{1}{g+h_0}\big)\big(\mathcal{M}(g) + \mathcal{M}(h)\big)} w_{xx} + w = f, \quad x \in (0,1) \\ w'(0) = w'(1) = 0. \end{cases}$$

- Oscilaciones rápidas en la frontera inferior y superior, $(\beta > 1)$.

$$\begin{cases} -\dfrac{g_0 + h_0}{\mathcal{M}(g) + \mathcal{M}(h)} w_{xx} + w = f, \quad x \in (0,1), \\ w'(0) = w'(1) = 0. \end{cases}$$

$\mathcal{M}(\cdot)$ denota el valor medio de la función.

Finalmente, en la Sección 3.4 analizamos el caso donde los dominios finos presentan oscilaciones débiles en ambas fronteras, superior e inferior. Aunque los resultados son nuevos, las técnicas usadas son muy similares a las aplicadas en [7].

El prototipo de dominio fino considerado en esta sección se corresponde con (0.3.4) donde $0 < \alpha, \beta < 1$. Entonces, probamos que el problema límite de (0.1.1) en este caso particular viene dado por

$$\begin{cases} -\dfrac{p_0}{\mathcal{M}(g)+\mathcal{M}(h)} u_{0xx} + u_0 = f, & x \in (0,1), \\ u_0'(0) = u_0'(1) = 0, \end{cases}$$

donde la constante p_0 es tal que

$$\frac{1}{h\left(\frac{x}{\epsilon^\alpha}\right) + g\left(\frac{x}{\epsilon^\beta}\right)} \overset{\epsilon\to 0}{\rightharpoonup} \frac{1}{p_0} \quad w - L^2(0,1),$$

y p_0 es dada por

$$\frac{1}{p_0} = \begin{cases} \lim\limits_{T\to\infty} \dfrac{1}{T} \displaystyle\int_0^T \dfrac{1}{g(y)+h(y)}\, dy, & \text{if } \alpha = \beta, \\ \dfrac{1}{L_1 L_2} \displaystyle\int_0^{L_1} \int_0^{L_2} \dfrac{1}{g(y)+h(z)}\, dz dy, & \text{if } \alpha \neq \beta. \end{cases}$$

0.4. Conclusiones

Los resultados obtenidos en esta tesis nos permiten dar un paso más en la compresión de cómo la geometría del domino a nivel microscópico afecta al comportamiento de las soluciones. Alejándonos de las hipótesis clásicas de periodicidad hemos sido capaces de analizar dominios finos con fronteras localmente periódicas, tanto la amplitud como el periodo de las oscilaciones varían en espacio, y también dominios finos con diferentes frecuencias y perfiles de oscilación en la frontera superior e inferior.

Además, aunque en este trabajo nos centramos en sentar las bases matemáticas de los métodos propuestos y en entender cómo la geometría de los dominios afecta al problema límite más que en las aplicaciones, creemos que las técnicas presentadas pueden contribuir a clarificar cómo tratar otro tipo de problemas más interesantes desde el punto de vista físico definidos en estructuras que de alguna manera presentan un comportamiento no periódico similar al de los dominios considerados en esta tesis.

Finalmente, nos gustaría decir que el trabajo recogido en esta memoria ha generado nuevas e interesantes cuestiones en la misma línea de investigación: extensión a dimensiones superiores, analizar el comportamiento de los correspondientes problemas parabólicos no lineales, tasa de los errores de convergencia de las soluciones, discusión del interés numérico de los métodos, etc.

Introduction

Thin domains, that is, domains where one or several of their characteristic directions are substantially smaller than the others, appear in many fields of science, like fluid dynamics (lubrication, conduction of fluids in thin tubes, ocean dynamics...), solid mechanics (thin rods, plates or shells) or even physiology (blood circulation). Thus, the wide possibilities of applying the mathematical results to real situations has made that partial differential equations on thin domains becomes a very studied topic over the last years.

From a mathematical point of view, the study of the solutions of a PDE on thin domains is a particular case of the general question concerning the effects of the variation in domains on the behavior of the solutions of the PDE. In this framework, obtaining the limit equation of the model considered on the thin domain, comparing the limit solution and the solutions of the equation defined on the thin domain, analyzing the coefficients of the limit equation and understanding how the geometry of the thin domains affects the limit equation are some of the main goals that should be reached. In fact, answering this kind of questions not only provide important theoretical results, it is also very interesting for the applications. For instance, being able to reduce the original problem to an easier to handle limit problem, which reflects most of important features of the original one is very useful for engineers and applied scientists.

In this thesis, we consider thin domains in dimension 2 with rough boundaries. This setting presents the special feature that besides the thickness of the domain another small parameter plays an important role in the description of the limit system: it is the small period of the oscillations. This way, obtaining the limit equation is more delicate due to the possible interaction between the different microscales.

The study of problems in thin domains with oscillating boundaries is not new. There are several works in the literature on the subject. We will mention some of them here and we also refer to their corresponding bibliographies. In [83, 84, 85] the authors study the asymptotic behavior of solutions to certain elliptic and parabolic problems in a thin perforated domain with rapidly varying thickness. The results obtained in these papers are related to the construction of a suitable asymptotic expansion of the solutions. This approach was proposed by T. Melnyk in [82] for the investigation of elliptic and spectral problems in thin perforated domains with rapidly varying thickness.

In the context of fluid flows we would like to cite [38] where the asymptotic behavior of the solutions of the Navier–Stokes system in a thin domain with a rough boundary is studied, see also the references therein. Other interesting papers studying the effect of rough boundaries on the behavior of fluid flows are [4, 29, 66, 76, 77].

There are several papers dedicated to the study of thin structures with a so called "rough boundary" from the point of view of the elasticity. For example, [25, 26] are devoted to describe the asymptotic behavior of an elastic multistructure composed of a set of periodic elastic rods in junction with a thin plate. We also refer the reader to [5, 16, 28] where the asymptotic description of nonlinearly elastic thin films with a fast-oscillating profile was obtained.

Other interesting works dealing with equations in domains with oscillating boundaries are [18, 41, 42] where complete asymptotic expansions of the solutions were studied.

Thus, the purpose of this thesis is to analyze the behavior of the solutions of certain elliptic equations in thin domains with rough boundaries beyond the classical periodic setting. More precisely, we will consider thin domains with a locally periodic oscillatory boundary and thin domains where both boundaries, top and bottom, present oscillations but the order of frequency and the profile of the oscillation is not the same at the top and at the bottom. We are particularly interested in understanding how the micro geometry of the domain affects the behavior of the solution. Hence, we will derive the limit problem of a model equation in different thin domains with oscillatory boundaries and we will investigate the effects of the micro-scale structure upon the macroscopic behavior of the solutions.

This is precisely the aim of the mathematical theory of homogenization. The theory of homogenization studies the macro-behavior of a model (typically a partial differential equation or an energy functional) which is "microscopically" heterogeneous in order to describe some characteristics of the heterogeneous medium. For instance, the homogenization theory deals with partial differential equations with coefficients that oscillate depending on a small scale, say ϵ, or with partial differential equations defined in domains whose geometry depends strongly on the small parameter ϵ like perforated domains (see e.g. [51, 50, 43, 78, 68]) or domains with oscillatory boundaries (see e.g. [33, 5, 22, 23]). The origin of this word is related to the question of replacement of the heterogenous material by an "equivalent" homogenous one. Roughly speaking, from the mathematical point of view this means that the solutions of a boundary value problem, depending on a small parameter, converge in an appropriate sense to the solution of a limit boundary value problem which is explicitly described.

The homogenization theory is very interesting from the point of view of the applications, for instance, it plays an important role in the mathematical analysis of physical, chemical and mechanical phenomena in strongly heterogeneous materials like composites, perforated media, rough media, porous media and similar situations. In this direction many works can be found in the literature, among others, we refer the reader to [64, 54, 55, 56] where some interesting chemical processes in porous media are studied, to the book [75] which contains some chapters devoted to different physical phenomena of flow and transport through porous media, to [69] where the authors address the homogenization of a problem for the Laplace operator arising, for example, in modeling diffusion of substances in perforated media with large adsorption parameters on the boundary of the perforations and to [37] where is analyzed the behavior of a viscous fluid in rough media.

Four branches of the homogenization theory can be distinguished: the G or H-convergence which places no restriction on the size of arrangement of the heterogeneities (see e.g. [90, 91, 105, 107, 104]); the second one deals with probabilistic and stochastic descriptions of heterogeneous media (see e.g. [20, 94]); the third one is De Giorgi's Γ-convergence [63, 62, 58] suitable for homogenization of optimization problem; the fourth one is devoted to periodic structures (see e.g. [19, 101, 52]).

Various methods have been developed in the context of periodic homogenization. For instance, asymptotic expansion methods together with the method of oscillatory test functions of Tartar (see [19, 101, 44]), the method of two-scale convergence [2, 92], which can be interpreted as an intermediate convergence between weak and strong convergence in L^p and has the capability to capture rapid oscillations on a prescribed fine-scale. Recently, the periodic unfolding method was introduced by D. Cioranescu, A. Damlamian and G. Griso (see [45, 46]), it is related to the dilation technique (see [6, 34]).

Notice that the methods devised to tackle problems in periodic structures are very powerful although less general than others. After all, periodicity is a very strong assumption. These methods have been developed for a long time and they have proved to be very efficient showing important advantages. First of all, periodicity is the easiest framework to work with. For instance, the two-scale asymptotic expansion method allows us to obtain the homogenized limit problem and even error estimates in an easy way under the assumption of periodicity. Second, they allow us to analyze very complicated models which are not simple to treat using other methods. Moreover, as we will show in this thesis, their importance goes far beyond the classical periodic setting. In particular, in this thesis we show how adapting methods originally devised to deal with periodic structures we can analyze more general situations: locally periodic oscillatory boundaries where both amplitude and period of the oscillations varying in space and thin domains with two oscillatory boundaries with different profiles and frequency of oscillation.

Before giving a detailed and complete outline of this thesis we would like to mention some papers which motivated this work. The works [7, 8, 9, 10] deal with different classes of oscillating thin domains discussing limit problems and properties of convergence. Let us describe in detail the main results obtained in these works in order to better understanding the context of this thesis.

To do so, we present here a very interesting model problem which allows us to avoid additional technical difficulties and to fix the general ideas. Thus, using the results introduced in [7, 8, 9, 10] we analyze in this introduction the asymptotic behavior of the solutions of the following Neumann problem for the Laplace operator

$$\begin{cases} -\Delta w^\epsilon + w^\epsilon = f & \text{in } R^\epsilon, \\ \dfrac{\partial w^\epsilon}{\partial \nu^\epsilon} = 0 & \text{on } \partial R^\epsilon, \end{cases} \tag{0.4.1}$$

where $f \in L^2(0,1)$, ν^ϵ is the unit outward normal to ∂R^ϵ and R^ϵ is a two dimensional oscillating thin domain with order of thickness ϵ defined as follows

$$R^\epsilon = \Big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ 0 < y < \epsilon\, G_\epsilon(x)\Big\}, \tag{0.4.2}$$

with $G_\epsilon(\cdot)$ oscillating as the parameter ϵ tends to zero. To simplify the presentation of the results we assume that $G_\epsilon(\cdot)$ is of the type

$$G_\epsilon(x) = a(x) + g(x/\epsilon^\alpha) \tag{0.4.3}$$

where $\alpha > 0$ is a parameter, $a : \mathbb{R} \to \mathbb{R}$ is a smooth positive function and $g : \mathbb{R} \to \mathbb{R}$ is an $L-$periodic smooth function satisfying $0 < g_0 \leq g(\cdot) \leq g_1$ for some fixed positive constants g_0 and g_1.

Observe that, the existence and uniqueness of solutions for problem (0.4.1) are guaranteed by Lax-Milgram Theorem for every fixed $\epsilon > 0$.

Notice also that the behavior of the solutions will depend essentially on the value of the parameter α. Moreover, since the domain R^ϵ has order of thickness ϵ it is expected that the family of solutions w^ϵ will converge to a function of just one variable as the parameter ϵ tends to zero, that is, the function limit will not depend on the "macroscopic" variable y.

First of all, we would like to point that it is well known that if the thin domain does not present oscillations $G_\epsilon(x) = G(x)$, say $g \equiv 0$ or $\alpha \equiv 0$ in (0.4.3), the limit equation is given by

$$\begin{cases} -\dfrac{1}{G(x)}\big(G(x) w_x(x)\big)_x + w(x) = f, \quad x \in (0,1), \\ \qquad\qquad w_x(0) = w_x(1) = 0, \end{cases}$$

see for example [73, 100] for details. The standard way to prove this result is to perform a change of variables, $x_1 = x$, $x_2 = \epsilon\, G(x) y$, which transforms the thin domain R^ϵ into a fixed reference domain, $R = (0,1)\times(0,1)$, and injects the parameter ϵ into the coefficients of the differential operators of the problem. After that, it is not difficult to obtain the limit problem using the weak formulation of the differential equations.

The techniques presented in [7] in order to study the behavior of the eigenvalues and eigenfunctions on thin domains with an oscillatory boundary may be easily adapted to obtain the limit of (0.4.1) when the thin domain presents weak oscillations, that is, $0 < \alpha < 1$ in (0.4.3). Thus, taking into account that

$$\begin{aligned} G_\epsilon(x) = a(x) + g\Big(\frac{x}{\epsilon^\alpha}\Big) &\overset{\epsilon\to 0}{\rightharpoonup} m(x) := a(x) + \frac{1}{L}\int_0^L g(s)\, ds \quad w - L^2(0,1), \\ \frac{1}{G_\epsilon(x)} = \frac{1}{a(x) + g\Big(\frac{x}{\epsilon^\alpha}\Big)} &\overset{\epsilon\to 0}{\rightharpoonup} k(x) := \frac{1}{L}\int_0^L \frac{1}{a(x) + g(s)}\, ds \quad w - L^2(0,1), \end{aligned}$$

then, the limit problem is

$$\begin{cases} -\dfrac{1}{m(x)}\Big(\dfrac{1}{k(x)} w_x\Big)_x + w = f, \quad x \in (0,1), \\ \qquad\qquad w_x(0) = w_x(1) = 0. \end{cases}$$

See Chapter 4 of [7] for details and some generalizations. The proof of this result consists of reducing the 2-dimensional original problem to a 1-dimensional problem with highly oscillating diffusion in which it is not difficult to pass to the limit. It is worth remarking that such technique works if and only if

$$\lim_{\epsilon \to 0} \epsilon \, G'_\epsilon(x) \stackrel{\epsilon \to 0}{\longrightarrow} 0, \text{ uniformly in } (0,1),$$

which is true in our example provided that $0 < \alpha < 1$.

The authors in [8, 9] deal with a class of thin domains that cover the case $\alpha = 1$ in (0.4.3). Notice that this situation is very resonant since the amplitude and period of the oscillations are of the same order ϵ, which also coincide with the order of the height of the thin domain. Thus, this scaling makes the determination of the limit problem not straightforward.

Specifically, the purely periodic case, say $G_\epsilon(x) = g(x/\epsilon)$ in (0.4.3), was studied in [8], see also [84], where the authors analyze not only the elliptic equation but also the corresponding semilinear parabolic problem. Observe that, this is a problem within the classical periodic setting since the oscillatory boundary is given by an $L-$periodic function. In fact, there is a representative cell describing the geometry of the domain, which is defined as follows

$$Y^* = \{(y_1, y_2) \in \mathbb{R}^2 \,:\, 0 < y_1 < L, \; 0 < y_2 < g(y_1)\}.$$

Thus, in [8] the limit problem was obtained using standard techniques in homogenization theory, specially those related to reticulated structures, we refer to [19, 101, 44, 104] for an exposition of the classical homogenization theory and [52] for more concrete developments in reticulated structures. In particular, the authors formally get the limit equation through the multiple-scale method and then they use the technique of oscillating test functions of Tartar, see [105, 106], to prove the convergence theorem. At this point it is worth observing that, as in the case of perforated domains, it is necessary to construct an extension operator since the geometry of the domain R^ϵ depends strongly on ϵ, and moreover, there is no inclusion relation between the spaces $L^2(R^\epsilon)$ and $L^2(R^{\epsilon'})$ when $\epsilon \neq \epsilon'$.

Then, the limit problem of (0.4.1) in this particular case is

$$\begin{cases} -q_0 w_{xx} + w = f(x), \quad x \in (0,1), \\ \qquad w'(0) = w'(1) = 0 \end{cases}$$

where

$$q_0 = \frac{1}{|Y^*|} \int_{Y^*} \Big\{ 1 - \frac{\partial X}{\partial y_1}(y_1, y_2) \Big\} dy_1 dy_2,$$

and X is the unique solution (up to constants) which is L-periodic in the first variable, of the problem:

$$\begin{cases} -\Delta X = 0 \text{ in } Y^*, \\ \dfrac{\partial X}{\partial N} = 0 \text{ on } B_2, \\ \dfrac{\partial X}{\partial N} = -\dfrac{g'(y_1)}{\sqrt{1 + g'(y_1)^2}} \text{ on } B_1, \end{cases}$$

where B_1 is the upper boundary and B_2 is the lower boundary of Y^*.

Let us remark that the geometry of the domain enters the limit equation through the diffusion coefficient. In fact, the diffusion coefficient depends on X which is the convenient auxiliary harmonic function defined in the representative basic cell Y^* which depends on the function $g(\cdot)$.

On the other hand, a generalization beyond the periodic case was addressed in [9]. In this paper the authors include the case where the amplitude of the oscillations depends on the point in a continuous way. For instance, taking into account our model of thin domain (0.4.2) we may think that the function which defines the oscillatory boundary is given by (0.4.3) with $\alpha = 1$. The limit problem for this case is given by

$$\begin{cases} -\dfrac{1}{p(x)}(q(x)w_x)_x + w = f(x), \quad x \in (0,1), \\ w'(0) = w'(1) = 0, \end{cases}$$

where

$$q(x) = \int_{Y^*(x)} \Big\{1 - \frac{\partial X(x)}{\partial y_1}(y_1, y_2)\Big\} dy_1 dy_2,$$
$$p(x) = |Y^*(x)|,$$

and $X(x)$ is the unique solution which is L-periodic in the first variable, of the problem

$$\begin{cases} -\Delta X(x) = 0 \text{ in } Y^*(x), \\ \dfrac{\partial X(x)}{\partial N} = 0 \text{ on } B_2(x), \\ \dfrac{\partial X(x)}{\partial N} = N_1(x) \text{ on } B_1(x), \\ \displaystyle\int_{Y^*(x)} X(x)\, dy_1 dy_2 = 0, \end{cases}$$

where $B_1(x)$ is the upper boundary and $B_2(x)$ is the lower boundary of the representative cell $Y^*(x)$ given by

$$Y^*(x) = \{(y_1, y_2) \in \mathbb{R}^2 \;:\; 0 < y_1 < L, \quad 0 < y_2 < a(x) + g(y_1)\}, \; \forall x \in (0,1).$$

Note that this case contains the previous one since we can recover the limit problem of the purely periodic case assuming that $a(\cdot) \equiv 0$. Moreover, observe that the dependence on x is explicitly stated in the limit as would be expected. Indeed, the homogenized coefficient of the limit equation depends on a family of auxiliary solutions posed in a one-parameter family of representative cells which capture in some way the locally periodic geometry of the thin domain.

It is worth to notice that none of the techniques used to solve the previous cases apply here. Hence, in [9] the authors propose a very interesting new method combining concepts and techniques from the homogenization theory with a domain perturbation result. The authors solve first the piecewise periodic case and then, they do an approximation argument to obtain the appropriate limit problem. With respect to the perturbation result, we would like to remark that they analyze in detail how the solutions of (0.4.1) depend on the thin domain, in particular, they prove a continuos dependence result with respect to the function G_ϵ.

In addition, the question of correctors for these two resonant problems, $\alpha = 1$ in (0.4.3), was addressed in [96, 97]. Applying the corrector approach developed by Bensoussan, Lions and Papanicolaou in [19], the authors get strong convergences when the original solutions are replaced by the first-order expansion through the Multiple-Scale Method. Moreover, they give error estimates for the purely periodic case. We refer the reader to [19, 44, 93] for a classical introduction to correctors approach and error estimates in periodic homogenization problems.

Finally, in the recent article [10] the authors study the case where the thin domain presents an extremely high oscillatory behavior at the boundary, that is, $\alpha > 1$ in (0.4.3). Here the roughness is so strong that there is a natural division of the domain into two parts

$$\begin{aligned} R^\epsilon_+ &= \Big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ \epsilon\, G_0(x) < y < \epsilon\, a(x) + \epsilon\, g(x/\epsilon^\alpha)\Big\}, \\ R^\epsilon_- &= \Big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ 0 < y < \epsilon\, G_0(x)\Big\}. \end{aligned}$$

where for every $x \in (0,1)$ the function G_0 is defined as follows

$$G_0(x) = \min_{y_1 \in \mathbb{R}} \{a(x) + g(y_1)\} = a(x) + g_0.$$

This way, in order to obtain the limit problem the authors analyze the properties of the solutions in each part and, consequently, they construct suitable test functions which allows to pass to the limit. The homogenized equation is given by

$$\begin{cases} -\dfrac{L}{\int_0^L \big(a(x) + g(y)\big)\, dy} \Big(\big(a(x) + g_0\big) w_x\Big)_x + w = f(x), \quad x \in (0,1) \\ \qquad\qquad w'(0) = w'(1) = 0. \end{cases}$$

Therefore, it is clear that the contributions of the papers cited above solve important issues for the general problem of modeling the asymptotic behavior of solutions of a partial differential equation on thin domains with an oscillatory boundary. However, several important questions arise at this point:

- Is it possible to obtain the same limit problems assuming less regularity on the oscillatory boundary of the thin domain?
- What are the correctors functions which involve strong convergences for the case where the thin domain presents weak oscillations? And for the thin domain with an extremely high oscillatory boundary?
- How does the presence of two oscillatory boundaries affects the asymptotic behavior of the solutions?
- In the context of locally periodic thin domains, what is the influence of a variable period on the macroscopic behavior of the solutions?

- What is the homogenized limit problem for the situation where the thin domain presents two oscillatory boundaries?
- Are the proof of the homogenization results for doubly oscillatory thin domains simple adaptations of the known techniques or present extra difficulties?

Hence, our motivation is to answer these questions and to increase our understanding of the linear elliptic theory in problems defined in oscillating thin domains. In the following we give a detailed outline of this thesis.

In Chapter 1 we adapt the unfolding operator method introduced in [45] by D. Cioranescu, A. Damlamian and G. Griso to the study of periodic thin domains with an oscillatory boundary. We refer to [59, 46] for a further detailed description of the method applied to the classical homogenization problem. The unfolding method has been successfully applied to many problems. For instance, in [48] this technique was used to study the homogenization in domains with holes, in [71, 72] the author obtains error estimates for periodic homogenization, the unfolding method was also used to study different problems defined in domains with oscillating boundaries (see e.g. [60, 24, 25, 26, 38]) and for other interesting applications see [70, 47, 65, 11, 49]. One of the main ideas of this method is to introduce the so-called unfolding operator (similar to the dilation operator, see [6, 34, 35, 36]), based on a change of variables which allows one to obtain the homogenization results with minimal hypothesis on the regularity of the structures. Moreover, we would like to point out that this approach is strongly related to the two-scale convergence method introduced by G. Nguetseng in [92], and further developed by G. Allaire in [2]. For periodic homogenization, the two-scale convergence of a sequence of functions is equivalent to the weak convergence of the corresponding unfolded sequence. Notice that, the unfolding operator method bypass the difficulties encountered in other methods, in particular the two-scale convergence method, with respect to the space and topology of test functions. As a matter of fact, the unfolding method only deals with classical notions of convergence in L^p spaces.

Then, the purpose of this chapter is to introduce the unfolding operator method as a general approach which allows us to obtain homogenization results for periodic thin domains with an oscillatory boundary. The sections of this chapter are devoted to examine the behavior of the solutions of the Poisson's equation with homogeneous Neumann boundary conditions posed in the different purely periodic thin domains, that is, we consider thin domains defined as follows

$$R^\epsilon = \Big\{(x, y) \in \mathbb{R}^2 \mid x \in (0, 1),\ 0 < y < \epsilon\, g(x/\epsilon^\alpha)\Big\},$$

where $\alpha > 0$ and $g : \mathbb{R} \to \mathbb{R}$ is an $L-$periodic function satisfying $0 < g_0 \leq g(\cdot) \leq g_1$ for some fixed positive constants g_0 and g_1. In particular, in Section 1.1 we introduce the unfolding operator for this kind of thin domains and we show its main properties while in the following three sections we obtain the homogenized limit problem and new correctors results depending on the order of the frequency of the oscillations.

We point out that the unfolding operator developed here provides an easy technique for homogenization of partial differential equations in thin domains with a periodic oscillatory boundary. Moreover, it is very important to point out that this method allows us to replace the regularity of the oscillatory boundary required in previous works, see [7, 8], by the existence of a Poincaré−Wirtinger inequality in the representative cell. Therefore we may consider a larger class of thin domains, with non-smooth boundaries, see introduction of Chapter 1 for details.

In addition, we believe that this first chapter will allow the reader to get a better understanding of the next chapter where the unfolding method is extended to a thin domain with a locally periodic oscillatory boundary.

Chapter 2 deals with the same kind of partial differential equations as the previous chapter, but now the structure considered is a 2-dimensional thin domain with a locally periodic oscillatory boundary. This means that beyond the periodic case both the amplitude and the period of the oscillations may not be constant and actually they vary in space. For instance, although throughout the chapter we will consider more general cases, we can assume in order to state the general ideas in this Introduction, that the thin domain is given by

$$R^\epsilon = \big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\, 0 < y < \epsilon\, \big(a(x) + g(\frac{x}{l(x)\epsilon})\big)\big\}, \tag{0.4.4}$$

where $a : (0,1) \to \mathbb{R}$ is a bounded smooth function, $g : \mathbb{R} \to \mathbb{R}$ is a 1-periodic smooth function and $l : \mathbb{R} \to \mathbb{R}$ is certain positive smooth function.

Observe that, since the periodicity properties vary from point to point in $x \in (0,1)$ properly speaking there is not a basic cell associated to the domain R^ϵ. However, by analogy with the periodic case we will refer to the following domain

$$Y^*(x) = \{(y_1, y_2) \in \mathbb{R}^2 \ : \ 0 < y_1 < l(x), \quad 0 < y_2 < a(x) + g\big(y_1/(l(x))\}, \tag{0.4.5}$$

as the basic cell at x.

Notice also that our setting includes the case analyzed in [9], locally periodic thin domains with constant period, but it also includes the most interesting case where both amplitude and period of the oscillations vary as we vary x. Moreover, we want to point out that this problem is not a simple generalization of the results in [9]. On one hand, it seems very delicate to extend the approach introduced in [9] to the present situation where the period is not constant. On the other hand, the way in which the variable period affects to the homogenized limit equation is not clear a priori. For instance, it is not obvious whether the limit equation of (0.4.1) where R^ϵ is given by (0.4.4) should be

$$-\frac{1}{|Y^*(x)|}\big(r(x)w_x\big)_x + w = f, \qquad \text{or} \qquad -\frac{l(x)}{|Y^*(x)|}\Big(r(x)\Big(\frac{1}{l(x)}w\Big)_x\Big)_x + w = f,$$

where $r(x) = \displaystyle\int_{Y^*(x)} \Big\{1 - \frac{\partial X(x)}{\partial y_1}(y_1,y_2)\Big\}dy_1dy_2$, or maybe

$$-\frac{l(x)}{|Y^*(x)|}\big(r(x)w_x\big)_x + w = f,$$

with $r(x) = \dfrac{1}{l(x)} \displaystyle\int_{Y^*(x)} \left\{1 - \frac{\partial X(x)}{\partial y_1}(y_1, y_2)\right\} dy_1 dy_2$, or maybe other. Observe that all these equations coincide if we consider a constant period.

Hence, the purpose of this work is to provide mathematical tools that contribute to analyze the influence of the locally periodic micro structures in the macro properties of the system. Notice that it is very interesting from a point of view of the application to real problems. As we have mentioned, thin structures with oscillating boundaries appear in many fields of science as fluid dynamics (lubrication), solid mechanics (thin rods, plates or shells) or even physiology (blood circulation). These thin structures are particular cases of the more extense subject of reticulated or perforated domains, which is very relevant in composite materials. Modelling these real world phenomena and understanding the effects of the microstructure in the macro properties of the material is a real challenge. In a first approximation we may assume that at the micro level, the material is completely periodic. The mathematical theory of homogenization of periodic structures, see [19, 52, 101, 44], provides a tool to analyze the phenomena in this case. Nevertheless in many practical situations the purely periodic case is not completely realistic and these structures present periodicity properties at the micro scale which may vary at a macro scale and therefore they are different at distinct points of the material, see for instance [31, 53, 99, 103] for different real world problems in a non purely periodic situation. This results in a locally periodic structure where the basic cell repeats itself approximately periodic in a small neighborhood of each point in space but its periodicity properties (shape of the cavity or even the period in which the cell is repeated) may be very different at two distinct points in space (at the macro level). In the particular case of thin structures this may be modelled as in our case, assuming that the function G, which describes the thin domain, has a dependence on the macro variable x affecting the shape and the period at the micro scale. Hence, providing mathematical tools that will contribute to analyze the influence of this locally periodic micro structures in the macro properties is interesting.

As a matter of fact, although the amount of research in the field of homogenization is quite large, only a small number of works on homogenization in locally periodic structures exist in the literature. We would like to mention some of them. In [39, 88, 89] an asymptotic expansion technique was used to obtain the limit problem and the estimates of the rate of the convergence for problems defined in domains with locally periodic perforations, i.e. the geometry of the cavities varies with space. Two scale convergence was applied in [40, 79] to homogenize the warping, the torsion and the Neumann problems in two dimensional domains with smooth changing holes and in [99] two scale convergence was generalized to a locally periodic and fibrous media. In [32] the author studies the homogenization of the conductivity problem defined in non-periodic and locally-periodic domains consisting of spherical balls. The method of H$-$convergence developed by Murat-Tartar, see [91], was used in order to obtain the macroscopic equations. Estimates for the numerical approximation of this problem were obtained in [102]. In [1] and [80] some non-periodic perforated structures were studied. In [31] the author describes the global behavior of three models of non-periodic fibrous materials providing a homogenization result for each case. Optimization of elastic bodies featuring a locally periodic microscopic pattern was considered in [17].

Despite the works mentioned above, the case introduced in this chapter has not been treated previously in the literature. In order to accomplish our goal and obtain the homogenized limit problem for the general "locally periodic" situation where not only the amplitude but also the period of the oscillations depends on the spatial variable x we extend the Unfolding Operator method. At this point, it is worth observing that the construction of the new unfolding operator is not trivial and the properties of this operator are not a simple adaptation of the equivalent results for the periodic case. We refer to Sections 2.2 and 2.3 to see in detail the delicate differences respect to the unfolding operator introduced in the previous chapter for the periodic models.

In Sections 2.4 and 2.5 we apply the results of the previous sections to get the associated homogenized limit problem, together with a corrector result. Then, if the thin domain is given by (0.4.4) we prove that the limit equation equation of (0.4.1) is

$$\begin{cases} -\dfrac{l(x)}{|Y^*(x)|}\big(r(x)w_x\big)_x + w = f, \quad x \in (0,1), \\ \qquad\qquad w'(0) = w'(1) = 0, \end{cases} \tag{0.4.6}$$

where

$$r(x) = \frac{1}{l(x)} \int_{Y^*(x)} \Big\{1 - \frac{\partial X(x)}{\partial y_1}(y_1, y_2)\Big\} dy_1 dy_2,$$

and $X(x,\cdot,\cdot)$ is the unique solution (up to constants) which is $l(x)$-periodic in the first variable of the corresponding auxiliary problem defined in the basic cell $Y^*(x)$ given by (0.4.5).

Notice that in case the period is constant, $l(x) \equiv L$, we recover the homogenized limit problem obtained in [9] using an approximation argument. The main results of this chapter were announced in [13] for the simpler and more intuitive locally periodic cases and they are written in [15].

Furthermore, we would like to point out that this extension to certain non-periodic structures is further evidence of the versatility and potential of the unfolding operator method to solve homogenization problems. We believe that our extension of the method to a locally periodic setting with varying period may shed light in how to deal with some other related problems in different situations.

Chapter 3 is dedicated to the study of the behavior of solutions to the Poisson's equation with homogeneous Neumann boundary conditions posed in a thin domain with doubly oscillatory boundary. Indeed, we think that this is a very natural way to extend the models studied in Chapter 1 to a more realistic situations where several microscopic scales appear.

More precisely, we deal in this chapter with thin domains which presents oscillations of amplitude ϵ on both boundaries, top and bottom. As a matter of fact, the model thin domain is given by

$$R^\epsilon = \Big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ -\epsilon\, h(x/\epsilon^\alpha) < y < \epsilon\, g(x/\epsilon^\beta)\Big\}, \quad \text{with } \beta, \alpha > 0. \tag{0.4.7}$$

where $g, h : \mathbb{R} \to \mathbb{R}$ are C^1 periodic functions with period L_1 and L_2 respectively. Moreover, there exist constants $h_0 \geq 0$ and $h_1, g_0, g_1 > 0$ such that $0 \leq h_0 \leq h(\cdot) \leq h_1$, and $0 < g_0 \leq g(\cdot) \leq g_1$.

In this framework, we are interested in analyzing in detail how the limiting model captures the different oscillatory behavior at the boundary. Notice that, as main novelty, we allow that the upper and lower boundary present different orders of frequency and profiles of oscillation. Therefore, we may distinguish six different behaviors of the oscillatory boundary depending on the parameters α and β: resonant and fast oscillations ($\beta = 1, \alpha > 1$), weak and fast oscillations ($\beta < 1, \alpha > 1$), fast oscillations at the top and the bottom boundary ($\alpha, \beta > 1$), doubly weak oscillatory boundary ($\alpha, \beta < 1$), resonant and weak oscillations ($\beta = 1, \alpha < 1$) and resonant case ($\alpha = \beta = 1$). In this chapter we study the first four model cases. Nowadays, we are analyzing the remaining two cases to complete the study of thin domains with doubly oscillatory boundary. Let us point out that the techniques introduced in this chapter are not a simple generalization of the methods applied to thin domains with only one oscillatory boundary. In fact, in this chapter we show in detail how to combine suitably different techniques to obtain the homogenized limit problem and some correctors results for the diverse cases.

More specifically, in Section 3.1 we combine classical methods in periodic homogenization and results of perturbations in order to analyze the case where the thin domain presents a fast oscillatory behavior at the lower boundary and oscillations at the upper boundary with the same order of frequency as the thickness of the domain, $\beta = 1$ and $\alpha > 1$ in the model thin domain (0.4.7). Thus, we obtain the homogenized limit problem and establish important convergence properties for the solutions. Therefore, the limit equation corresponding to problem (0.4.1) with the prototype thin domain (0.4.7) with $\beta = 1$ and $\alpha > 1$ is given by

$$\begin{cases} -\dfrac{\hat{q}}{\frac{|Y^*|}{L_1} + p} w_{xx} + w = f(x), \quad x \in (0,1) \\ w'(0) = w'(1) = 0 \end{cases} \tag{0.4.8}$$

where

$$\hat{q} = \frac{1}{L_1} \int_{Y^*} \Big\{1 - \frac{\partial X}{\partial y_1}(y_1, y_2)\Big\} dy_1 dy_2, \quad p = \frac{1}{L_2} \int_0^{L_2} h(s) ds - h_0,$$

and X is a convenient auxiliary harmonic function defined in the basic cell Y^* corresponding to the upper oscillatory boundary.

Notice that homogenized limit problem reflects the oscillatory behavior of the thin domain. As a matter of fact, in the diffusion coefficient one may distinguish clearly two kind of terms: $\hat{q}$ from the resonant oscillations and p from the extremely high oscillatory boundary.

We refer to Subsection 3.1.1 and Subsection 3.1.2 for a detailed description of the results. Furthermore, the question of correctors is addressed in Subsection 3.1.3. We provide a new corrector result, the construction of the corrector function is not standard, which allows to deduce relevant information on how the oscillatory profile of the domain affects the behavior of the solutions. Some of the main results of this chapter were presented in [12].

To conclude this section we generalize the previous results to certain perforated thin domains with doubly oscillatory boundary, see Subsection 3.1.4.

The following two sections complete the study of thin domains with at least one fast oscillatory boundary. In fact, we combine the unfolding operator method introduced in Chapter 1 with the choice of appropriate oscillating test functions to obtain the homogenized limit problem for the cases where the thin domain presents a fast oscillatory behavior at the bottom boundary and a weak oscillatory behavior at the top boundary, $\alpha > 1$ and $0 < \beta < 1$ in (0.4.7), and where the thin domain presents two fast oscillatory boundaries, $\alpha, \beta > 1$ in (0.4.7).

In particular Section 3.2 is devoted to the case with fast and weak oscillations, $\alpha > 1$ and $\beta < 1$, and Section 3.3 to the case with two fast oscillatory boundaries, $\alpha, \beta > 1$. Therefore, we prove that the homogenized limit problem associated to problem (0.4.1) with the prototype thin domain (0.4.7) is given by

- Fast and slow boundary oscillations, $(0 < \beta < 1)$.

$$\begin{cases} -\dfrac{1}{\mathcal{M}\big(\frac{1}{g+h_0}\big)\big(\mathcal{M}(g)+\mathcal{M}(h)\big)} w_{xx} + w = f, \quad x \in (0,1) \\ w'(0) = w'(1) = 0. \end{cases} \tag{0.4.9}$$

- Fast oscillations at the top and the bottom boundary, $(\beta > 1)$.

$$\begin{cases} -\dfrac{g_0+h_0}{\mathcal{M}(g)+\mathcal{M}(h)} w_{xx} + w = f, \quad x \in (0,1), \\ w'(0) = w'(1) = 0. \end{cases} \tag{0.4.10}$$

$\mathcal{M}(\cdot)$ denotes the mean value of the function.

Finally, in Section 3.4 we analyze the case where the thin domain presents weak oscillations on both boundaries, top and bottom. Although the results are new, they cannot be found in the literature, the techniques are very similar to the ones applied in [7].

Thus, the prototype thin domain we consider in this section is given by (0.4.7) with $0 < \alpha, \beta < 1$. Then, we will prove that the limit problem of (0.4.1) in this particular case is given by

$$\begin{cases} -\dfrac{p_0}{\mathcal{M}(g)+\mathcal{M}(h)} u_{0xx} + u_0 = f, \quad x \in (0,1), \\ u_0'(0) = u_0'(1) = 0, \end{cases} \tag{0.4.11}$$

where the constant p_0 is such that

$$\frac{1}{h\left(\frac{x}{\epsilon^\alpha}\right)+g\left(\frac{x}{\epsilon^\beta}\right)} \overset{\epsilon\to 0}{\rightharpoonup} \frac{1}{p_0} \quad w-L^2(0,1), \tag{0.4.12}$$

and p_0 is given by

$$\frac{1}{p_0}=\begin{cases} \lim\limits_{T\to\infty}\dfrac{1}{T}\displaystyle\int_0^T \frac{1}{g(y)+h(y)}\,dy, & \text{if } \alpha=\beta,\\ \dfrac{1}{L_1L_2}\displaystyle\int_0^{L_1}\int_0^{L_2}\frac{1}{g(y)+h(z)}\,dzdy, & \text{if } \alpha\neq\beta. \end{cases}$$

We conclude this introduction by mentioning that the methods presented in this thesis could be applied to other problems of greater interest from a physical point of view. However, here we focused on the mathematical foundations of the methods and on accomplishing a deep understanding of how the geometry of the domains affects the limiting problem rather than its applications. Moreover, we would like to point out that this thesis has instigated new interesting questions in the same research direction such as: extensions to higher dimensions, analyzing the behavior of the solutions of the corresponding nonlinear parabolic problems, rates of convergence of the solutions, discussion of the numerical interest of the methods, etc.

Part of the results of this thesis have been communicated in the following scientific publications: the results from Subsections 3.1.1 and 3.1.2 have been published in [12]; some results of Chapter 1 and Section 3.2 have appeared in [13]; the results from Chapter 2 have been announced in [14] for the most intuitive case and they have been described in [15] for the general case.

All the results have been presented at various international and national scientific meetings, such as "XXIV CEDYA" congress in Cádiz (2015), "Séminaire GT Calcul des variations et GT Homogénéisation et échelles multiples" in the Laboratoire Jacques-Louis Lions (Paris, 2014), "Mini-courses in Mathematical Analysis 2014" in the University of Padova (2014), "10th AIMS Conference on Dynamical Systems Differential Equations and Applications" in Madrid (2014), "ICMC Summer Meeting On Differential Equations 2014" in Sao Carlos (Brazil 2014), "XXIII CEDYA" congress in Castellón (2013), "ICMC Summer Meeting On Differential Equations 2013" in Sao Carlos (Brazil 2013).

Furthermore, three papers have been published outside the framework of this thesis: G. Griso, M. Villanueva-Pesqueira, "Straight rod with different order of thickness", Asymptotic Analysis, 94, 3-4 (2015), pp. 255 - 291; Juan J. Nieto, Rosana Rodríguez-Lopez, Manuel Villanueva-Pesqueira, "Exact solution to the periodic boundary value problem for a first-order linear fuzzy differential equation with impulses", Fuzzy Optimization and Decision Making, 10 - 4 (2011), 323 - 339; Juan J. Nieto, Rosana Rodríguez-Lopez, Manuel Villanueva-Pesqueira "Green's Function for the Periodic Boundary Value Problem Related to a First-order Impulsive Differential Equation and Applications to Functional Problems", Differ. Equ. Dyn. Syst.19 - 3 (2011), pp. 199 - 210.

Notation

Before embarking into the statements and proofs of the results let us clarify some notation that we will use throughout this thesis.

- We employ the letter C to denote any constant independent of small parameters. The exact value denoted by C may therefore change from line to line in a given computation.
- R^ϵ: A thin domain with order of thickness ϵ.
- Y^*: A reference cell.
- $\tilde{}$ usually denotes the standard extension by zero.
- $|\Omega|$: The Lebesgue measure of Ω.
- $\overline{\Omega} \equiv$ closure of Ω. $\mathrm{Int}(\Omega) \equiv$ interior of Ω.
- $I = (0, 1)$.
- $\mathcal{M}_\Omega(\phi)$: The mean value (or the average) of a function $\phi \in L^1(\Omega)$ over Ω

$$\mathcal{M}_\Omega(\phi) = \frac{1}{|\Omega|} \int_\Omega \phi(x)\, dx.$$

- The restriction of a function $\phi : \Omega \to \mathbb{R}$ to a subset $U \subset \Omega$ is denoted by $\phi|_U$.

Function spaces

Let $\Omega \subset \mathbb{R}^n$ be an open set.

- $C^k(\Omega) = \{\varphi : \Omega \to \mathbb{R} : \varphi$ is k-times continuously differentiable$\}$.
- $\mathcal{D}(\Omega)$ or $\mathcal{C}_0^\infty(\Omega)$: The space of indefinitely differentiable functions with compact support in Ω.
- $L^p(\Omega) = \{\varphi : \Omega \to \mathbb{R} : \varphi$ is Lebesgue measurable, $\|\varphi\|_{L^p(\Omega)} < \infty\}$, where

$$\|\varphi\|_{L^p(\Omega)} < \infty\} = \Big(\int_\Omega |\varphi|^p dx \Big)^{1/p} \quad (1 \leq p < \infty).$$

  $L^\infty(\Omega) = \{\varphi : \Omega \to \mathbb{R} : \varphi$ is Lebesgue measurable, $\|\varphi\|_{L^\infty(\Omega)} < \infty\}$, where

$$\|\varphi\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |\varphi|.$$

- The Sobolev space $W^{k,p}(\Omega)$ consists of all locally summable functions $\varphi : \Omega \to \mathbb{R}$ such that for each multi-index α with $|\alpha| \leq k$ the $D^\alpha(\varphi)$ exists in the weak sense and belongs to $L^p(\Omega)$. That is,

$$W^{k,p}(\Omega) = \left\{ \varphi \in L^p(\Omega) \middle| \begin{array}{c} \forall \alpha \text{ with } |\alpha| \leq k, \exists g_\alpha \in L^p(\Omega) \text{such that} \\ \int_\Omega \varphi D^\alpha \psi = (-1)^{|\alpha|} \int_\Omega g_\alpha \psi \quad \forall \psi \in \mathcal{C}_0^\infty(\Omega), \end{array} \right\}$$

where we use the standard multi-index notation $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ with $\alpha_i \geqslant 0$ an integer,

$$|\alpha| = \sum_{i=1}^n \alpha_i \quad \text{ and } D^\alpha \psi = \frac{\partial^{|\alpha|}\psi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}.$$

If $p = 2$ we write $H^k(\Omega) = W^{k,p}(\Omega)$.

- Due to the order of the height of the thin domains R^ϵ it makes sense to consider the following measure in the thin domains

$$\rho_\epsilon(\mathcal{O}) = \frac{1}{\epsilon}|\mathcal{O}|, \; \forall \, \mathcal{O} \subset R^\epsilon.$$

The rescaled Lebesgue measure ρ_ϵ allows us to preserve the relative capacity of a measurable subset $\mathcal{O} \subset R^\epsilon$. Moreover, using the previous measure we introduce the spaces $L^p(R^\epsilon, \rho_\epsilon)$ and $W^{1,p}(R^\epsilon, \rho_\epsilon)$, for $1 \leq p < \infty$ endowed with the norms obtained rescaling the usual norms by the factor $\frac{1}{\epsilon}$, that is,

$$|||\varphi|||_{L^p(R^\epsilon)} = \epsilon^{-1/p}||\varphi||_{L^p(R^\epsilon)}, \quad \forall \varphi \in L^p(R^\epsilon),$$
$$|||\varphi|||_{W^{1,p}(R^\epsilon)} = \epsilon^{-1/p}||\varphi||_{W^{1,p}(R^\epsilon)}, \quad \forall \varphi \in W^{1,p}(R^\epsilon).$$

It is very common to consider this kind of norms in works involving thin domains, see e.g. [73, 100, 98, 96].

- The subindex # denotes periodicity, as is common in homogenization theory. We will use it to denote periodicity with respect to the first variable of the reference cell. For instance, if $Y^* = (0, L) \times (0, g_0)$, the space $C_\#\big(Y^*\big)$ consists of all functions φ which are obtained as restrictions to Y^* of functions in $C(\mathbb{R}^2)$ which are L-periodic in the first variable. That is

$$C_\#\big(Y^*\big) = \{\varphi_{|Y^*} : \varphi \in C(\mathbb{R}^2), \, \varphi(y_1 + L, y_2) = \varphi(y_1, y_2), \quad \forall (y_1, y_2) \in \mathbb{R}^2\}.$$

- Consider also the Banach spaces $L^p\Big((0,1); W^{1,q}_\#\big(Y^*\big)\Big)$, $L^p\Big((0,1); C^k_\#\big(Y^*\big)\Big)$ and $W^{1,p}\Big((0,1); C^k_\#\big(Y^*\big)\Big)$ in a standard way. For instance, the Banach space $L^p\Big((0,1); W^{1,q}_\#\big(Y^*\big)\Big)$ consists of the measurable functions $\varphi : (0,1) \times Y^* \to \mathbb{R}$, such that $\varphi(x,\cdot,\cdot) \in W^{1,q}_{\#}\big(Y^*\big)$ a.e. $x \in (0,1)$, with

$$\|\varphi\|_{L^p\Big((0,1);W^{1,q}_{\#}\big(Y^*\big)\Big)} := \begin{cases} \Big(\displaystyle\int_0^1 \|\varphi(x,\cdot,\cdot)\|^p_{W^{1,q}_{\#}\big(Y^*\big)}\, dx\Big)^{1/p} < \infty, & \text{for } 1 \le p < \infty \\ \displaystyle\operatorname*{ess\,sup}_{x\in(0,1)} \|\varphi(x,\cdot,\cdot)\|_{W^{1,q}_{\#}\big(Y^*\big)} < \infty, & \text{for } p = \infty. \end{cases}$$

In addition, we use the standard notation for the quotient space $W^{1,q}_{\#}\big(Y^*\big)/\mathbb{R}$, that is the quotient of $W^{1,q}_{\#}\big(Y^*\big)$ modulo the constants functions, and then, we also consider $L^p\Big((0,1);W^{1,q}_{\#}\big(Y^*\big)/\mathbb{R}\Big)$.

Chapter 1

Unfolding method in thin domains with an oscillatory boundary

In this chapter we provide a comprehensive presentation of the periodic unfolding method adapted to thin domains with a periodic oscillatory boundary. We present the method in this particular situation and show that it provides a general approach to analyze in a systematic and unified way, elliptic problems posed on these domains. We will choose the prototype of elliptic problem given by the standard Laplace operator with Neumann boundary conditions.

Our thin domain will always have order of thickness $\epsilon > 0$ and is defined as follows

$$R^\epsilon = \Big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ 0 < y < \epsilon\, g(x/\epsilon^\alpha)\Big\}, \tag{1.0.1}$$

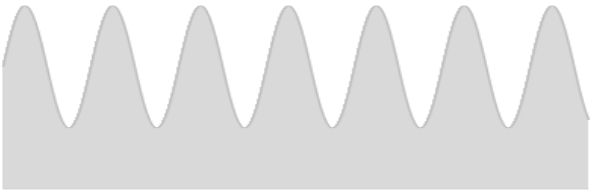
where $\alpha > 0$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is an $L-$periodic function satisfying $0 < g_0 \leq g(\cdot) \leq g_1$ for some fixed positive constants g_0 and g_1. Moreover, in the whole section we will assume that the function g is not a constant function in order to guarantee the oscillating behavior of the upper boundary.

Observe that the different values of $\alpha > 0$ will give us different types of oscillatory behavior or rugosity at the boundary. We will distinguish three different cases, see Figure 1.1. More precisely:

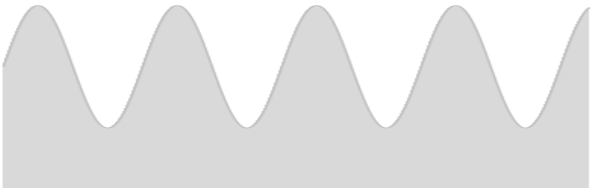
- $0 < \alpha < 1$. We will refer to this case as “weak oscillatory” case. The order of the period of the oscillations is ϵ^α, which is much larger than the order of the amplitude of the oscillations, ϵ, or the order of the thickness of the domain, also ϵ. Notice that in this case, if the function g is smooth enough, then the function $x \to \epsilon g(x/\epsilon^\alpha)$ is uniformly $C^{1,\theta}$ for some $\theta < 1 - \alpha$ and it goes to zero in $C^{1,\theta}(\mathbb{R})$.

- $\alpha = 1$. We will refer to this case as “resonant” or “critical” case. Notice that the order of the period coincides with the order of the amplitude of the oscillations and it also coincides with the order of the thickness of the domain. Moreover, if again the function g is smooth enough, then the function $x \to \epsilon g(x/\epsilon)$ is uniformly C^1 but it does not go to 0 in this topology.

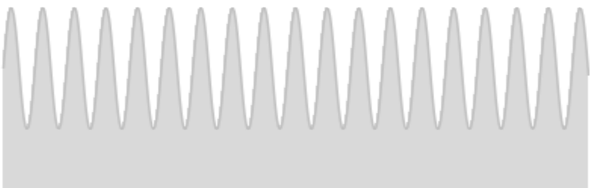
- $\alpha > 1$. We will refer to this case as “fast” or “extremely high oscillatory” case. The order of the period of the oscillations is much smaller than the order of the amplitude of the oscillations or the order of the thickness of the domain. The function $x \to \epsilon g(x/\epsilon^{\alpha})$ is uniformly bounded in some Hölder norm but not in C^1.



(a) Resonant case, $\alpha = 1$



(b) Weak oscillations, $0 < \alpha < 1$



(c) Extremely high oscillatory boundary, $\alpha > 1$

Figure 1.1: Thin domains defined by the same function g with different order of frequency of the oscillations.

One of the basic ideas of the unfolding method is to define an operator, which will be denoted as $\mathcal{T}_\epsilon$ and is named “unfolding operator”, that transforms functions defined in R^ϵ into functions defined in a fixed domain W in such a way that the oscillations of the domain and of the functions defined in R^ϵ are naturally “unfolded” into the new domain W. Having the functions now defined on a fixed domain, will allow us to pass to the limit in appropriate norms. A way to accomplish this for the thin domain (1.0.1) is as follows.

Since the function g is L-periodic, we define the “basic cell”

$$Y^* = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \in (0, L),\ 0 < y_2 < g(y_1)\}.$$

and observe that the domain R^ϵ is obtained by “repetition” of the following rescaling of the basic cell:

$$Y^*_\epsilon = \{(\epsilon^\alpha y_1, \epsilon y_2) : (y_1, y_2) \in Y^*\}.$$

Actually, if we define the points

$$x_k^\epsilon = kL\epsilon^\alpha, \quad k = 0, 1, 2, \dots N_\epsilon$$

where $N_\epsilon = 1/(L\epsilon^\alpha)$ (and we may assume in this introduction to simplify that we take ϵ so that $1/(L\epsilon^\alpha)$ is an integer) then R^ϵ is given by

$$R^\epsilon = \bigcup_{k=0}^{N_\epsilon - 1} (x_k^\epsilon + Y_\epsilon^*)$$

Hence, if seems natural to define the set in $\mathbb{R}^3$

$$W = (0,1) \times Y^* = \{(x, y_1, y_2) : 0 < x < 1, (y_1, y_2) \in Y^*\}$$

Then, a function u^ϵ defined in R^ϵ is transformed into a function U^ϵ defined in W intuitively as follows: we transform the part of the function u^ϵ defined in $x_k^\epsilon + Y_\epsilon^*$ into the part of the function U^ϵ defined in the set $\{x_k\} \times Y^*$ with the appropriate rescaling in the (y_1, y_2) variables. Doing this for all the sets $x_k^\epsilon + Y_\epsilon^*$, $k = 0, 1, \dots N^\epsilon - 1$ gives us the function U^ϵ defined in the set

$$\{x_0^\epsilon, x_1^\epsilon, \dots, x_{N_\epsilon - 1}^\epsilon\} \times Y^* \subset (0,1) \times Y^*$$

To define the function U^ϵ for all $x \in (0,1)$ we just repeat the definition of the function in $x_k^\epsilon \times Y^*$ to all $x \times Y^*$, with $x \in [x_k^\epsilon, x_{k+1}^\epsilon)$. That is

$$U^\epsilon(x, y_1, y_2) = U^\epsilon(x_k^\epsilon, y_1, y_2), \quad \forall x \in [x_k^\epsilon, x_{k+1}^\epsilon)$$

and

$$U^\epsilon(x_k^\epsilon, y_1, y_2) = u^\epsilon(x_k^\epsilon + \epsilon^\alpha y_1, \epsilon y_2)$$

Another way to see the unfolding operator is as the pullback of the map:

$$\begin{aligned} S_\epsilon : W &\longrightarrow R^\epsilon \\ (x, y_1, y_2) &\longrightarrow ([x]_\epsilon + \epsilon^\alpha y_1, \epsilon y_2) \end{aligned}$$

where $[x]_\epsilon = x_k^\epsilon$ if $x \in [x_k^\epsilon, x_{k+1}^\epsilon)$. That is, $U^\epsilon = u_\epsilon \circ S_\epsilon$. The map that transforms u_ϵ into U^ϵ will be denoted by $\mathcal{T}_\epsilon$, that is $U^\epsilon = \mathcal{T}_\epsilon(u^\epsilon)$.

This operator will allow us also to transform integrals defined in R^ϵ into integrals defined in W and in particular we will transform the weak formulations of elliptic problems in R^ϵ into expressions involving integrals in W. The fact that W is a fixed domain will ease up the pass to the limit. In particular, we will treat the prototype elliptic problem in the thin domain R^ϵ,

$$\begin{cases} -\Delta u^\epsilon + u^\epsilon = f^\epsilon & \text{in } R^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial \nu^\epsilon} = 0 & \text{on } \partial R^\epsilon \end{cases} \tag{1.0.2}$$

where $f^\epsilon \in L^2(R^\epsilon)$, ν^ϵ is the unit outward normal to ∂R^ϵ, for which the variational formulation is: find $u^\epsilon \in H^1(R^\epsilon)$ such that

$$\int_{R^\epsilon} \Big\{ \frac{\partial u^\epsilon}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial u^\epsilon}{\partial y}\frac{\partial \varphi}{\partial y} + u^\epsilon \varphi \Big\}\, dxdy = \int_{R^\epsilon} f^\epsilon \varphi \, dxdy, \quad \forall \varphi \in H^1(R^\epsilon). \tag{1.0.3}$$

Hence, we will treat this problem for the three different types of thin domains described above using the unfolding operator method.

We would like to point out that this kind of problems have already been discussed in previous works using other techniques. As we have mentioned in the Introduction, the case where the thin domain presents weak roughness ($0 < \alpha < 1$) was treated in [7], the resonant case ($\alpha = 1$) was studied in [8, 84] using standard techniques in homogenization and the problem on thin domains with a fast oscillatory boundary ($\alpha > 1$) was analyzed in [10]. However, this is the first time that the three cases are tackled in a unified way using the unfolding operator method.

We show how the unfolding operator method applies in the three cases with very mild assumptions on the regularity of the domain R^ϵ, which is related to the regularity of the function g. As a matter of fact, since the method does not require a extension procedure, we will be able to deal with complicated geometries on R^ϵ. For instance, we may admit thin domains where the function g is continuous, comb-like thin domains or domains where extension operators do not apply, see Figure 1.2. Our general requirements for the function g are expressed in hypothesis ($\mathbf{H_g}$) in Section 1.1.

However we would like to point out that everything is more complicated if we consider thin domains beyond the periodic setting as we will see in Chapter 2.

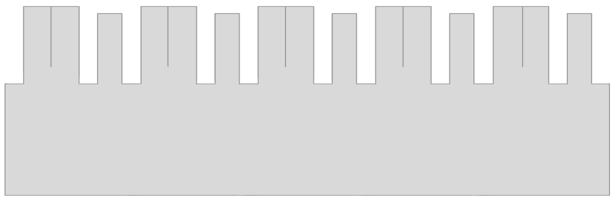
As we have already mentioned in the Introduction, the unfolding method has been successfully applied to other problems to study the effect of rough boundaries on the behavior of the solution of partial differential equations. Among others papers, we can cite [60] for an application of the method to a 2-dimensional domain with oscillating boundaries (actually [60] is the first time where the unfolding operator method is applied to a domain with an oscillatory boundary). More precisely, using the periodic unfolding method, the homogenized limit problem and a result of strong convergence is obtained for a variational problem on a sequence of 2-dimensional domains with oscillating boundaries. In the framework of the linearized elasticity we would like to mention [24, 25, 26] where the authors combine a technique based on the appropriate decomposition of the displacement field with the unfolding method to describe the homogenization process for the junction of rods and a plate. More recently, in [38] the authors using the unfolding method to study the asymptotic behavior of the solutions of the Navier-Stokes system in a thin domain satisfying the Navier boundary condition on a periodic rough set.

This chapter is organized in four sections.

- In Section 1.1, we introduce the unfolding operator and we prove its main properties in thin domains without any extra regularity condition. We also introduce the averaging operator which is the adjoint of the unfolding operator. We complete this section with several relevant convergence results.

(a) Domain with continuous boundary

(b) Comb-like domain



(c) It is not possible to build an extension operator

Figure 1.2: Examples of thin domains.

- Section 1.2 is dedicated to the resonant case which was studied in [8, 84] using extension operators. We apply the unfolding method to identify the homogenized limit problem replacing the existence of the extension operators by a Poincaré−Wirtinger hypothesis. In addition, we give a corrector result which makes use of the averaging operator.

- In Section 1.3 we study the case of weak oscillations in a similar way as the resonant case. We recover the limit problem obtained in [7] and we show how the unfolding operator allows us to easily get a new result of strong convergence, see Proposition 1.3.6.

- Section 1.4 concerns with the case of thin domains with very highly oscillatory boundaries. In such situations we show that the arguments used to get the convergence result for the previous two cases do not apply. Actually, we divide the thin domain in two different parts in a similar way to [10, 60], one of them contains the oscillating boundary and the other one is a non oscillating thin domain. We introduce a rescaling operator which allows to pass to the limit in the non oscillating part and we apply the unfolding operator introduced in Section 1.1 to the oscillating part of the domain. Furthermore, a new result of strong convergence is obtained.

Remark 1.0.1. *Part of the results of this chapter have appeared in [14].*

1.1. Definition of the unfolding operator and main properties

In this section we introduce the notation and the main properties of the unfolding operator in the case of thin domains. We will consider two-dimensional thin open sets with an oscillatory behavior in its top boundary which are a little more general than the ones described in the introduction and which are defined as follows

$$R^\epsilon = \Big\{(x, y) \in \mathbb{R}^2 \mid x \in (0, 1),\ \epsilon b < y < \epsilon\, g(x/\epsilon^\alpha)\Big\}, \tag{1.1.1}$$

where b is a positive constant, the parameters ϵ and α are greater than zero and the function $g : \mathbb{R} \to \mathbb{R}$ satisfies the following hypothesis

(Hg) $g : \mathbb{R} \to \mathbb{R}$ is defined for all $x \in \mathbb{R}$, it is $L-$periodic (that is $g(x + L) = g(x)\,\forall x \in \mathbb{R}$), it belongs to $L^\infty(\mathbb{R})$ and there exist two positive constants g_0 and g_1 such that $0 \le b \le g_0 \le g(x) \le g_1$ for all $x \in \mathbb{R}$, where $g_0 = \min_{x\in\mathbb{R}}\{g(x)\} \geqslant b$. Moreover, assume that $g(\cdot)$ is lower semicontinuous, that is, $g(x_0) \le \liminf\limits_{x\to x_0} g(x),\ \forall x_0 \in \mathbb{R}$.

The representative cell which describes the thin structure is given by

$$Y^* = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \in (0, L),\ b < y_2 < g(y_1)\}.$$

Remark 1.1.1. *Observe that from hypothesis **(Hg)** the representative cell Y^* is an open set. As a matter of fact, let (x_0, y_0) be a point in Y^*, then from the lower semicontinuity of the function $g(\cdot)$ we have*

$$y_0 < g(x_0) \le \liminf_{x\to x_0} g(x).$$

Thus, if we consider $\delta \equiv g(x_0) - y_0 > 0$ then, there exists $\epsilon > 0$ such that

$$g(x_0) - \delta/2 \le g(y_1) \quad \textit{for all } |y_1 - x_0| < \epsilon.$$

Therefore, for ϵ small enough we can guarantee that the neighborhood of the point (x_0, y_0) given by

$$U = \Big\{(y_1, y_2) \in \mathbb{R}^2 : |y_1 - x_0| < \epsilon, |y_2 - y_0| < \min\{\delta/2, (y_0 - b)/2\}\Big\}$$

is contained in Y^*.

Observe also that since Y^* is an open set we have that R^ϵ is an open set too.

Notice that the assumptions on the function g allows us to consider complex profiles of oscillation. For instance, if the function $g(\cdot)$ is piecewise periodic we obtain thin sets as ones depicted in Figure 1.2 and Figure 1.3.

(a) Model of basic cell Y^*

(b) Corresponding R^ϵ

Figure 1.3: Example of oscillating boundary

Furthermore, it is important to note that in our setting two functions $g(\cdot)$ and $\hat{g}(\cdot)$ satisfying that $g(x) = \hat{g}(x)$ for a.e. $x \in \mathbb{R}$ do not define the same basic cell Y^* and, as a consequence, the corresponding open sets R^ϵ are different too. For instance, consider the constant function $\hat{g} \equiv b + 2$ and the following L-periodic function

$$g(y_1) = \begin{cases} b+2 & \text{if } y_1 \in [0, L) \setminus \{L/2\}, \\ b+1 & \text{if } y_1 = L/2. \end{cases}$$

It is obvious that $g(x) = \hat{g}(x)$ for a.e. $x \in \mathbb{R}$ but, notice that, $\hat{g}$ defines the thin domain

$$\hat{R}^\epsilon = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, \epsilon b < y < \epsilon(b+2)\},$$

which does not present oscillations while g defines a thin domain $R^\epsilon = \hat{R}^\epsilon \setminus \bigcup I_k^\epsilon$ where I_k^ϵ is given by

$$I_k^\epsilon = \Big\{ \Big(\epsilon k L + \frac{\epsilon L}{2}, y \Big) : \epsilon(b+1) < y < \epsilon(b+2) \Big\},$$

where k is any integer satisfying $0 < \epsilon k L + \frac{\epsilon L}{2} < 1$, see Figure 1.4. Notice that, R^ϵ is a fissured media with very different properties from $\hat{R}^\epsilon$.

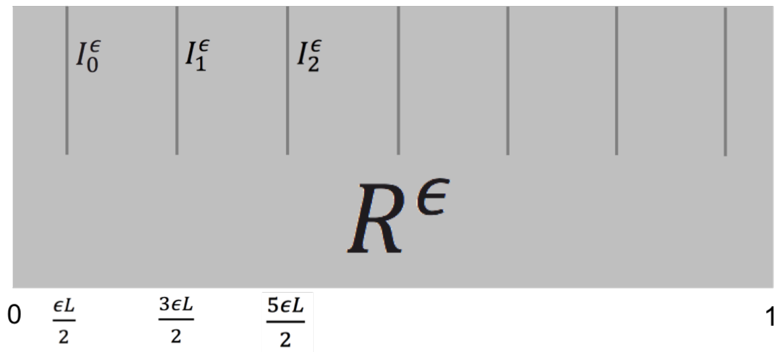


Figure 1.4: Fissured thin domain R^ϵ

Remark 1.1.2. *Note that the sets considered in this section may be disconnected, see Figure 1.5. Observe that the minimum of the function g which defines the oscillatory boundary can be equal to b.*

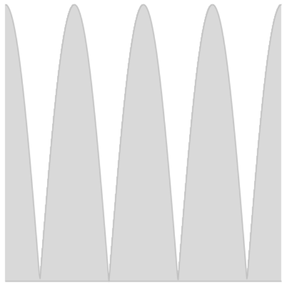


Figure 1.5: Disconnected R^ϵ

By analogy with the definition of the integer and fractional part of a real number, for $x \in \mathbb{R}$, $[x]_L$ denotes the unique integer such that $x \in \big[[x]_L L, ([x]_L + 1)L\big)$ and $\{x\}_L \in [0, L)$ is such that $x = [x]_L L + \{x\}_L$.

Then, for each $\epsilon > 0$ and for every $x \in \mathbb{R}$ there exists a unique integer, $\left[\frac{x}{\epsilon^\alpha}\right]_L$, such that

$$x = \epsilon^\alpha \left[\frac{x}{\epsilon^\alpha}\right]_L L + \epsilon^\alpha \left\{\frac{x}{\epsilon^\alpha}\right\}_L, \tag{1.1.2}$$

where $\left\{\frac{x}{\epsilon^\alpha}\right\}_L \in [0, L)$.

In addition, we will use the following notations

- $I^\epsilon = \mathrm{Int}\Big\{\bigcup_{k=0}^{N_\epsilon} [\epsilon^\alpha kL, \epsilon^\alpha L(k+1)]\Big\}$ where N_ϵ is the largest integer such that $\epsilon^\alpha L(N_\epsilon + 1) \leqslant 1$.
- $\Lambda^\epsilon = I \setminus I^\epsilon$, recall that $I = (0,1)$. Equivalently, $\Lambda^\epsilon = [\epsilon^\alpha L(N_\epsilon + 1), 1)$.

Observe that by construction Λ^ϵ converges in some sense to the empty set as ϵ goes to zero. Moreover, the set I^ϵ allows us to define R_0^ϵ, the set which contains all the cells totally included in R^ϵ

$$R_0^\epsilon = \Big\{(x,y) \in \mathbb{R}^2 \mid x \in I^\epsilon,\ \epsilon b < y < \epsilon\, g(x/\epsilon^\alpha)\Big\},$$

while Λ^ϵ gives us the subset of R^ϵ containing the corresponding part of the unique cell which is not totally included in R^ϵ, that is

$$R_1^\epsilon = \Big\{(x,y) \in \mathbb{R}^2 \mid x \in \Lambda^\epsilon,\ \epsilon b < y < \epsilon\, g(x/\epsilon^\alpha)\Big\}. \tag{1.1.3}$$

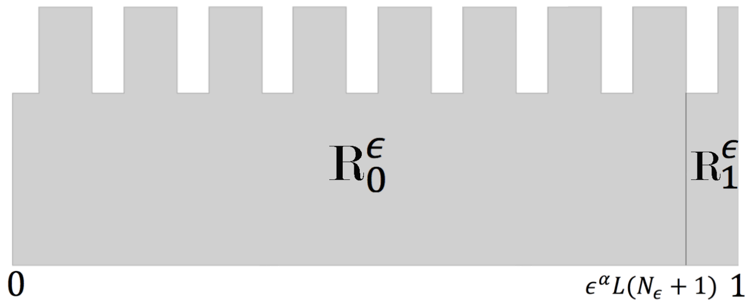


Figure 1.6: Sets R_0^ϵ and R_1^ϵ

The periodic unfolding operator is defined as:

Definition 1.1.3. *Let φ be a Lebesgue-measurable function in R^ϵ. The unfolding operator $\mathcal{T}_\epsilon$, acting on φ, is defined as the following function in* $(0,1) \times Y^*$

$$\mathcal{T}_\epsilon(\varphi)(x, y_1, y_2) = \begin{cases} \varphi\Big(\epsilon^\alpha \Big[\frac{x}{\epsilon^\alpha}\Big]_L L + \epsilon^\alpha y_1, \epsilon y_2\Big) & \textit{for} \quad (x, y_1, y_2) \in I^\epsilon \times Y^*, \\ 0 & \textit{for} \quad (x, y_1, y_2) \in \Lambda^\epsilon \times Y^*. \end{cases}$$

Note that the operator $\mathcal{T}_\epsilon$ transforms Lebesgue-measurable functions defined on R^ϵ into Lebesgue-measurable functions defined on the fixed set $(0,1) \times Y^*$ which are piecewise constant with respect to x. Therefore, if φ is very regular, $\mathcal{T}_\epsilon(\varphi)$ inherits the regularity as a function of (y_1, y_2) and not respect to the variable x.

As in classical periodic homogenization, the unfolding operator reflects two scales: the "macroscopic" scale x gives the position in the interval $(0,1)$ and the "microscopic" scale (y_1, y_2) gives the position in the cell Y^*. Notice that the oscillations of the boundary are captured in the variables (y_1, y_2) which belong to the basic cell Y^*.

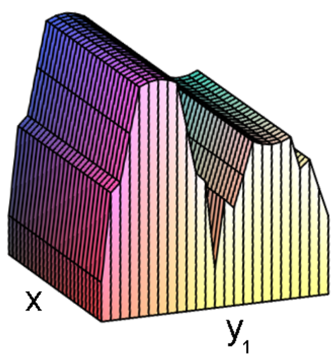


Figure 1.7: $(0,1) \times Y^*$ associated to the cell depicted in Figure 1.3

The following result considers several basic and somehow immediate properties of the unfolding operator.

Proposition 1.1.4. *The unfolding operator $\mathcal{T}_\epsilon$ has the following properties:*

i) $\mathcal{T}_\epsilon$ *is a linear operator.*

ii) $\mathcal{T}_\epsilon(\varphi\psi) = \mathcal{T}_\epsilon(\varphi)\mathcal{T}_\epsilon(\psi)$ $\forall\, \varphi, \psi$ *Lebesgue-measurable functions in* R^ϵ.

iii) Every function $\varphi \in L^p(R^\epsilon)$, *with* $1 \le p \le \infty$, *satisfies*

$$\mathcal{T}_\epsilon(\varphi)\Big(x, \Big\{\frac{x}{\epsilon^\alpha}\Big\}_L, \frac{y}{\epsilon}\Big) = \varphi(x,y), \quad \forall (x,y) \in R_0^\epsilon.$$

iv) Let φ *be a measurable function on* Y^* *extended by* $L-$*periodicity in the first variable. Then,* $\varphi^\epsilon(x,y) = \varphi(\frac{x}{\epsilon^\alpha}, \frac{y}{\epsilon})$ *is a measurable function on* R^ϵ *such that*

$$\mathcal{T}_\epsilon(\varphi^\epsilon)(x, y_1, y_2) = \varphi(y_1, y_2), \quad \forall (x, y_1, y_2) \in I^\epsilon \times Y^*.$$

Furthermore, if $\varphi \in L^p(Y^*)$, *with* $1 \le p \le \infty$ *then* $\varphi^\epsilon \in L^p(R^\epsilon)$.

v) Let $\varphi \in L^1(R^\epsilon)$. *The following integral equality holds*

$$\begin{aligned} &\frac{1}{L}\int_{(0,1)\times Y^*} \mathcal{T}_\epsilon(\varphi)(x,y_1,y_2)dxdy_1dy_2 = \frac{1}{\epsilon}\int_{R_0^\epsilon} \varphi(x,y)dxdy \\ &= \frac{1}{\epsilon}\int_{R^\epsilon} \varphi(x,y)dxdy - \frac{1}{\epsilon}\int_{R_1^\epsilon} \varphi(x,y)dxdy. \end{aligned}$$

vi) For every $\varphi \in L^p(R^\epsilon)$ *we have* $\mathcal{T}_\epsilon(\varphi) \in L^p\big((0,1)\times Y^*\big)$, *with* $1 \le p \le \infty$. *In addition, the following relationship exists between their norms:*

$$\|\mathcal{T}_\epsilon(\varphi)\|_{L^p\left((0,1)\times Y^*\right)} = \Big(\frac{L}{\epsilon}\Big)^{\frac{1}{p}} \|\varphi\|_{L^p(R_0^\epsilon)} \le \Big(\frac{L}{\epsilon}\Big)^{\frac{1}{p}} \|\varphi\|_{L^p(R^\epsilon)}.$$

In the special case $p = \infty$,

$$\|\mathcal{T}_\epsilon(\varphi)\|_{L^\infty\left((0,1)\times Y^*\right)} = \|\varphi\|_{L^\infty(R_0^\epsilon)} \le \|\varphi\|_{L^\infty(R^\epsilon)}.$$

vii) For every $\varphi \in W^{1,p}(R^\epsilon)$, $1 \le p \le \infty$, *one has*

$$\frac{\partial}{\partial y_1}\mathcal{T}_\epsilon(\varphi) = \epsilon^\alpha \mathcal{T}_\epsilon\Big(\frac{\partial \varphi}{\partial x}\Big) \;\; \textit{and} \;\; \frac{\partial}{\partial y_2}\mathcal{T}_\epsilon(\varphi) = \epsilon \mathcal{T}_\epsilon\Big(\frac{\partial \varphi}{\partial y}\Big),$$

a.e. for $(x, y_1, y_2) \in (0,1) \times Y^*$.

viii) If $\varphi \in W^{1,p}(R^\epsilon)$, *then* $\mathcal{T}_\epsilon(\varphi)$ *belongs to* $L^p\big((0,1); W^{1,p}(Y^*)\big)$, $1 \le p \le \infty$. *Moreover, the following relationship exists between their norms, in case* $1 \le p < \infty$

$$\begin{aligned} \Big\|\frac{\partial}{\partial y_1}\mathcal{T}_\epsilon(\varphi)\Big\|_{L^p\left((0,1)\times Y^*\right)} &= \epsilon^\alpha\Big(\frac{L}{\epsilon}\Big)^{\frac{1}{p}}\Big\|\frac{\partial \varphi}{\partial x}\Big\|_{L^p(R_0^\epsilon)} \le \epsilon^\alpha\Big(\frac{L}{\epsilon}\Big)^{\frac{1}{p}}\Big\|\frac{\partial \varphi}{\partial x}\Big\|_{L^p(R^\epsilon)}, \\ \Big\|\frac{\partial}{\partial y_2}\mathcal{T}_\epsilon(\varphi)\Big\|_{L^p\left((0,1)\times Y^*\right)} &= \epsilon\Big(\frac{L}{\epsilon}\Big)^{\frac{1}{p}}\Big\|\frac{\partial \varphi}{\partial y}\Big\|_{L^p(R_0^\epsilon)} \le \epsilon\Big(\frac{L}{\epsilon}\Big)^{\frac{1}{p}}\Big\|\frac{\partial \varphi}{\partial y}\Big\|_{L^p(R^\epsilon)}. \end{aligned}$$

For $p=\infty$ one has

$$\Big\|\frac{\partial}{\partial y_1}\mathcal{T}_\epsilon(\varphi)\Big\|_{L^\infty\big((0,1)\times Y^*\big)}=\epsilon^\alpha\Big\|\frac{\partial\varphi}{\partial x}\Big\|_{L^\infty(R_0^\epsilon)}\leq\epsilon^\alpha\Big\|\frac{\partial\varphi}{\partial x}\Big\|_{L^\infty(R^\epsilon)},$$
$$\Big\|\frac{\partial}{\partial y_2}\mathcal{T}_\epsilon(\varphi)\Big\|_{L^\infty\big((0,1)\times Y^*\big)}=\epsilon\Big\|\frac{\partial\varphi}{\partial y}\Big\|_{L^\infty(R_0^\epsilon)}\leq\epsilon\Big\|\frac{\partial\varphi}{\partial y}\Big\|_{L^\infty(R^\epsilon)}.$$

Proof. i) and ii) are a simple consequence of definition of the unfolding operator.

iii) Observe that if $(x,y)\in R_0^\epsilon$ then $\Big(x,\Big\{\frac{x}{\epsilon^\alpha}\Big\}_L,\frac{y}{\epsilon}\Big)\in I^\epsilon\times Y^*$. Therefore, from (1.1.2) and definition of the unfolding operator we have

$$\mathcal{T}_\epsilon(\varphi)\Big(x,\Big\{\frac{x}{\epsilon^\alpha}\Big\}_L,\frac{y}{\epsilon}\Big)=\varphi\Big(\epsilon^\alpha\Big[\frac{x}{\epsilon^\alpha}\Big]_L L+\epsilon^\alpha\Big\{\frac{x}{\epsilon^\alpha}\Big\}_L,y\Big)=\varphi(x,y),\quad\forall\varphi\in L^p(R^\epsilon).$$

iv) Let $(x,y)\in R^\epsilon$. It is obvious that there exists $k\in\mathbb{N}$ large enough such that

$$\frac{x}{\epsilon^\alpha}=y_1+kL.$$

Moreover, since $(x,y)\in R^\epsilon$ we have that $\epsilon b<y<\epsilon g\big(\frac{x}{\epsilon^\alpha}\big)=\epsilon g\big(\frac{x}{\epsilon^\alpha}-kL\big)$, where we use that $g(\cdot)$ is L-periodic.

As a consequence, we obtain

$$0<y_1<L\ \text{ and }\ b<y/\epsilon<g(y_1),$$

which implies $(y_1,y/\epsilon)\in Y^*$.

Therefore, for any L-periodic measurable function φ defined on Y^* we can define

$$\varphi^\epsilon(x,y)=\varphi\Big(\frac{x}{\epsilon^\alpha},\frac{y}{\epsilon}\Big),\ \forall(x,y)\in R^\epsilon.$$

Note that, using the periodic structure of R^ϵ and an obvious change of variables if $\varphi\in L^p(Y^*)$ we get $\varphi^\epsilon\in L^p(R^\epsilon)$ for $1\leq p<\infty$

$$\begin{aligned}\int_{R^\epsilon}|\varphi^\epsilon|^p\,dxdy&\leq\sum_{k=0}^{N_\epsilon+1}\int_{\epsilon^\alpha kL}^{\epsilon^\alpha L(k+1)}\int_{\epsilon b}^{\epsilon g(x/\epsilon^\alpha)}|\varphi^\epsilon|^p\,dydx\\&=\sum_{k=0}^{N_\epsilon+1}\epsilon^{\alpha+1}\int_{Y^*}|\varphi(y_1,y_2)|^p dy_1dy_2\\&=C\epsilon\int_{Y^*}|\varphi(y_1,y_2)|^p dy_1dy_2,\end{aligned}$$

where the constant does not depend on ϵ. The result is obvious for $p=\infty$.

Moreover, applying the unfolding operator to the oscillating function φ^ϵ we "almost recover" the initial function

$$\mathcal{T}_\epsilon(\varphi^\epsilon)(x,y_1,y_2)=\begin{cases}\varphi\Big(\Big[\frac{x}{\epsilon^\alpha}\Big]_L L+y_1,y_2\Big)=\varphi(y_1,y_2) & \text{for }(x,y_1,y_2)\in I^\epsilon\times Y^*,\\ 0 & \text{for }(x,y_1,y_2)\in\Lambda^\epsilon\times Y^*.\end{cases}$$

v) Suppose that $\varphi \in L^1(R^\epsilon)$. Then, using that $\mathcal{T}_\epsilon(\varphi^\epsilon)$ is piecewise constant in x it is easy to get the following integral equality which is fundamental for the unfolding method

$$\begin{aligned}
&\frac{1}{L}\int_{(0,1)\times Y^*} \mathcal{T}_\epsilon(\varphi)(x,y_1,y_2)\,dxdy_1dy_2\\
&= \frac{1}{L}\int_{I^\epsilon\times Y^*} \varphi\Big(\epsilon^\alpha\Big[\frac{x}{\epsilon^\alpha}\Big]_L L+\epsilon^\alpha y_1,\epsilon y_2\Big)\,dxdy_1dy_2\\
&= \frac{1}{L}\sum_{k=0}^{N_\epsilon}\int_{k\epsilon^\alpha L}^{(k+1)\epsilon^\alpha L}\int_{Y^*}\varphi\Big(\epsilon^\alpha\Big[\frac{x}{\epsilon^\alpha}\Big]_L L+\epsilon^\alpha y_1,\epsilon y_2\Big)\,dy_1dy_2dx\\
&= \frac{1}{L}\sum_{k=0}^{N_\epsilon}\int_{k\epsilon^\alpha L}^{(k+1)\epsilon^\alpha L}\int_{Y^*}\varphi\Big(\epsilon^\alpha kL+\epsilon^\alpha y_1,\epsilon y_2\Big)\,dy_1dy_2dx\\
&= \epsilon^\alpha\sum_{k=0}^{N_\epsilon}\int_{Y^*}\varphi\Big(\epsilon^\alpha kL+\epsilon^\alpha y_1,\epsilon y_2\Big)\,dy_1dy_2 = \frac{1}{\epsilon}\sum_{k=0}^{N_\epsilon}\int_{k\epsilon^\alpha L}^{(k+1)\epsilon^\alpha L}\int_{\epsilon b}^{\epsilon g(x/\epsilon^\alpha)}\varphi\,dydx\\
&= \frac{1}{\epsilon}\int_{R_0^\epsilon}\varphi(x,y)dxdy.
\end{aligned}$$

Then, the desired equality is straightforward.

vi) Let $\varphi \in L^p(R^\epsilon)$ for $1 \le p < \infty$. Then $|\varphi|^p \in L^1(R^\epsilon)$ and by properties ii) and v) we have

$$\begin{aligned}
\frac{1}{L}\int_{(0,1)\times Y^*}|\mathcal{T}_\epsilon(\varphi)|^p\,dxdy_1dy_2 &= \frac{1}{L}\int_{(0,1)\times Y^*}\mathcal{T}_\epsilon(|\varphi|^p)\,dxdy_1dy_2\\
&= \frac{1}{\epsilon}\int_{R_0^\epsilon}|\varphi|^p dxdy\\
&\le \frac{1}{\epsilon}\int_{R^\epsilon}|\varphi|^p dxdy.
\end{aligned}$$

The result for $p = \infty$ is straightforward.

vii) For any $\varphi \in W^{1,p}(R^\epsilon)$, from the definition of the unfolding operator, it follows that

$$\begin{aligned}
\nabla y_1 y_2 \mathcal{T}_\epsilon(\varphi) &= \Big(\epsilon^\alpha\frac{\partial\varphi}{\partial x},\epsilon\frac{\partial\varphi}{\partial y}\Big)\Big(\epsilon^\alpha\Big[\frac{x}{\epsilon^\alpha}\Big]_L L+\epsilon^\alpha y_1,\epsilon y_2\Big)\\
&= \Big(\epsilon^\alpha\mathcal{T}_\epsilon\Big(\frac{\partial\varphi}{\partial x}\Big),\epsilon\mathcal{T}_\epsilon\Big(\frac{\partial\varphi}{\partial y}\Big)\Big), \quad \text{a.e. for } (x,y_1,y_2)\in I^\epsilon\times Y^*,\\
\nabla y_1 y_2 \mathcal{T}_\epsilon(\varphi) &= (0,0), \quad \text{a.e. for } (x,y_1,y_2)\in \Lambda^\epsilon\times Y^*.
\end{aligned}$$

Therefore, we can conclude that

$$\nabla y_1 y_2 \mathcal{T}_\epsilon(\varphi) = \Big(\epsilon^\alpha\mathcal{T}_\epsilon\Big(\frac{\partial\varphi}{\partial x}\Big),\epsilon\mathcal{T}_\epsilon\Big(\frac{\partial\varphi}{\partial y}\Big)\Big), \text{ a.e. for } (x,y_1,y_2)\in(0,1)\times Y^*.$$

viii) This result is a immediate consequence of the properties vi) and vii).

□

Remark 1.1.5. *Notice that, due to the order of the height of the thin set the factor $\frac{1}{\epsilon}$ appears in properties v) and vi). As a matter of fact, from now on, we use the following rescaled norms in the thin open sets (see Notation Section)*

$$|||\varphi|||_{L^p(R^\epsilon)} = \epsilon^{-1/p}||\varphi||_{L^p(R^\epsilon)}, \quad \forall \varphi \in L^p(R^\epsilon), \quad 1 \le p < \infty,$$
$$|||\varphi|||_{W^{1,p}(R^\epsilon)} = \epsilon^{-1/p}||\varphi||_{W^{1,p}(R^\epsilon)}, \quad \forall \varphi \in W^{1,p}(R^\epsilon), \quad 1 \le p < \infty.$$

For completeness we may denote

$$|||\varphi|||_{L^\infty(R^\epsilon)} = ||\varphi||_{L^\infty(R^\epsilon)}.$$

In view of properties i) and vi), the unfolding operator $\mathcal{T}_\epsilon$ is linear and continuous from $L^p(R^\epsilon)$ to $L^p((0,1) \times Y^*)$ for $p \in [1, \infty)$. Moreover, for every $\varphi \in L^p(R^\epsilon)$ it satisfies

$$\|\mathcal{T}_\epsilon(\varphi)\|_{L^p\big((0,1)\times Y^*\big)} \le L^{1/p}\, |||\varphi|||_{L^p(R^\epsilon)},$$

Note that for $p = +\infty$ the unfolding operator is a linear isometry between the spaces $L^\infty(R_0^\epsilon)$ and $L^\infty((0,1) \times Y^*)$.

Property v) in Proposition 1.1.4 will be essential to pass to the limit when dealing with solutions of differential equations because it will allow us to transform any integral over the thin set depending on the parameter ϵ into an integral over the fixed set $(0,1) \times Y^*$. Notice that, in view of this property, if the thin open set may be written as the interior of the union of rescaled basic cells, that is $R^\epsilon = R_0^\epsilon$ or equivalently $\Lambda_\epsilon = \emptyset$, then the unfolding operator conserves, up to a factor $\frac{L}{\epsilon}$, the integral of the functions

$$\frac{1}{L}\int_{(0,1)\times Y^*} \mathcal{T}_\epsilon(\varphi)(x, y_1, y_2)dxdy_1dy_2 = \frac{1}{\epsilon}\int_{R^\epsilon} \varphi(x,y)dxdy.$$

However, in general one has $\Lambda_\epsilon \neq \emptyset$, so we say that this unfolding operator "almost preserves" the integral of the functions since the "integration defect" arises only from the unique cell which is not completely included in R^ϵ and it is controlled by the integral on R_1^ϵ. Therefore, an important concept for the unfolding method is the following property called unfolding criterion for integrals, u.c.i.

Definition 1.1.6. *A sequence $\{\varphi^\epsilon\}$ in $L^1(R^\epsilon)$ satisfies the unfolding criterion for integrals (u.c.i.) if*

$$\frac{1}{\epsilon}\int_{R_1^\epsilon} |\varphi^\epsilon| dxdy \xrightarrow{\epsilon\to 0} 0.$$

As an immediate consequence, if a sequence $\{\varphi^\epsilon\}$ satisfies the u.c.i we get from property v) of Proposition 1.1.4

$$\frac{1}{L}\int_{(0,1)\times Y^*} \mathcal{T}_\epsilon(\varphi^\epsilon)(x, y_1, y_2)dxdy_1dy_2 - \frac{1}{\epsilon}\int_{R^\epsilon} \varphi^\epsilon(x,y)dxdy \xrightarrow{\epsilon\to 0} 0.$$

In the next three propositions we obtain some criteria to guarantee that some functions satisfy the u.c.i.

Proposition 1.1.7. *Let φ^ϵ be a sequence in $L^p(R^\epsilon)$ for $p \in (1, \infty]$ with $|||\varphi^\epsilon|||_{L^p(R^\epsilon)}$ uniformly bounded. Then, it satisfies the unfolding criterion for integrals*

$$\frac{1}{\epsilon}\int_{R_1^\epsilon} |\varphi^\epsilon| dxdy \xrightarrow{\epsilon \to 0} 0.$$

Furthermore, let $\psi^\epsilon \in L^q(R^\epsilon)$ with $|||\psi^\epsilon|||_{L^q(R^\epsilon)}$ uniformly bounded for $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r}$ with $r > 1$. Then, the product sequence $\{\varphi^\epsilon \psi^\epsilon\}$ satisfies the unfolding criterion for integrals

$$\frac{1}{\epsilon}\int_{R_1^\epsilon} |\varphi^\epsilon \psi^\epsilon| dxdy \xrightarrow{\epsilon \to 0} 0.$$

Proof. First assume $1 < p < \infty$. Taking into account that $\rho_\epsilon(R_1^\epsilon) \xrightarrow{\epsilon \to 0} 0$, recall that ρ_ϵ denotes the rescaled Lebesgue measure ($\rho_\epsilon(\cdot) = \frac{1}{\epsilon}|\cdot|$ see Notation Section), and since there exists a constant C independent of ϵ such that $|||\varphi^\epsilon|||_{L^p(R^\epsilon)} < C$ we can ensure by Hölder's inequality that φ^ϵ satisfies the u.c.i.

$$\begin{aligned}
\frac{1}{\epsilon}\int_{R_1^\epsilon} |\varphi^\epsilon| dxdy &\le \frac{1}{\epsilon}||\varphi^\epsilon||_{L^p(R^\epsilon)} |R_1^\epsilon|^{\frac{1}{q}} \\
&= \epsilon^{-1/p}||\varphi^\epsilon||_{L^p(R^\epsilon)} \epsilon^{-1/q} |R_1^\epsilon|^{\frac{1}{q}} \\
&= |||\varphi^\epsilon|||_{L^p(R^\epsilon)} \rho_\epsilon(R_1^\epsilon)^{\frac{1}{q}} \xrightarrow{\epsilon \to 0} 0,
\end{aligned}$$

where q is such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

For $p = \infty$ we have that there exists a constant C independent of ϵ such that $||\varphi^\epsilon||_{L^\infty(R^\epsilon)} < C$. Then, we obtain

$$\begin{aligned}
\frac{1}{\epsilon}\int_{R_1^\epsilon} |\varphi^\epsilon| dxdy &\le ||\varphi^\epsilon||_{L^\infty(R^\epsilon)} \frac{1}{\epsilon}|R_1^\epsilon| \\
&\le C\rho_\epsilon(R_1^\epsilon) \xrightarrow{\epsilon \to 0} 0.
\end{aligned}$$

This proves the first statement.

For the second one, since $\varphi^\epsilon \psi^\epsilon \in L^r(R^\epsilon)$ for some $r > 1$ we obtain

$$\frac{1}{\epsilon}\int_{R_1^\epsilon} |\varphi^\epsilon \psi^\epsilon| dxdy \xrightarrow{\epsilon \to 0} 0.$$

□

Proposition 1.1.8. *Let $\varphi^\epsilon \in L^p(R^\epsilon)$ with $|||\varphi^\epsilon|||_{L^p(R^\epsilon)}$ uniformly bounded, $p \in (1, \infty]$, and $\phi \in L^q(0,1)$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then*

$$\frac{1}{\epsilon}\int_{R_1^\epsilon} |\varphi^\epsilon \phi| dxdy \xrightarrow{\epsilon \to 0} 0.$$

Proof. Observe that $\chi_{\Lambda^\epsilon}(x) \stackrel{\epsilon\to 0}{\longrightarrow} 0$ for all $x \in \mathbb{R}$. Consequently, taking into account that ϕ depends only on the variable x, the definition of R_1^ϵ and by Lebesgue's dominated convergence theorem we have

$$\frac{1}{\epsilon}\int_{R_1^\epsilon} |\phi|^q dxdy = \frac{1}{\epsilon}\int_{\Lambda^\epsilon}\int_{\epsilon b}^{\epsilon g(x/\epsilon^\alpha)} |\phi|^q dydx$$
$$= \int_{\Lambda^\epsilon} (g(x/\epsilon^\alpha) - b)|\phi|^q dx \le (g_1 - b)\int_{\mathbb{R}} |\phi|^q \chi_{\Lambda^\epsilon} dx \stackrel{\epsilon\to 0}{\longrightarrow} 0.$$

Hence, by Hölder's inequality we get the result

$$\frac{1}{\epsilon}\int_{R_1^\epsilon} |\varphi^\epsilon \phi| dxdy \le |||\varphi^\epsilon|||_{L^p(R_1^\epsilon)} \Big(\frac{1}{\epsilon}\int_{R_1^\epsilon} |\phi|^q dxdy\Big)^{1/q} \stackrel{\epsilon\to 0}{\longrightarrow} 0.$$

□

Proposition 1.1.9. *Let $\{\varphi^\epsilon\}$ be a uniformly bounded sequence in $L^p(R^\epsilon)$ for $1 < p < \infty$, $|||\varphi^\epsilon|||_{L^p(R^\epsilon)} \le C$, and let $\{\psi^\epsilon\}$ be the sequence in $L^q(R^\epsilon)$, $\frac{1}{p} + \frac{1}{q} = 1$, defined as follows*

$$\psi^\epsilon(x, y) = \psi(\frac{x}{\epsilon^\alpha}, \frac{y}{\epsilon})$$

where $\psi \in L^q(Y^*)$.

Then, the product sequence $\{\varphi^\epsilon \psi^\epsilon\}$ satisfies the unfolding criterion for integrals

$$\frac{1}{\epsilon}\int_{R_1^\epsilon} |\varphi^\epsilon \psi^\epsilon| dxdy \stackrel{\epsilon\to 0}{\longrightarrow} 0.$$

Proof. From the definition of R_1^ϵ and performing the same computations as in iv) of Proposition 1.1.4 we obtain

$$\frac{1}{\epsilon}\int_{R_1^\epsilon} |\psi^\epsilon|^q \, dxdy \le \frac{1}{\epsilon}\int_{\epsilon^\alpha L(N_\epsilon+1)}^{\epsilon^\alpha L(N_\epsilon+2)} \int_{\epsilon b}^{\epsilon g(x/\epsilon^\alpha)} \Big|\psi\big(\frac{x}{\epsilon^\alpha}, \frac{y}{\epsilon}\big)\Big|^q \, dydx$$
$$= \epsilon^\alpha \int_{Y^*} |\psi(y_1, y_2)|^q dy_1 dy_2.$$

Then, the sequence $\{\varphi^\epsilon \psi^\epsilon\}$ satisfies the u.c.i by Hölder's inequality

$$\frac{1}{\epsilon}\int_{R_1^\epsilon} |\varphi^\epsilon \psi^\epsilon| dxdy \le \epsilon^{-1/p} ||\varphi^\epsilon||_{L^p(R_1^\epsilon)} \Big(\frac{1}{\epsilon}\int_{R_1^\epsilon} |\psi^\epsilon|^q \, dxdy\Big)^{1/q} \le C\epsilon^{\alpha/q} \stackrel{\epsilon\to 0}{\longrightarrow} 0.$$

□

Now, we are going to analyze some convergence properties of the unfolding operator as ϵ goes to zero.

Proposition 1.1.10. *Let $\varphi \in L^p(0,1)$, $1 \le p < \infty$. Then considering φ as a function defined in R^ϵ we have*

$$\mathcal{T}_\epsilon(\varphi) \stackrel{\epsilon\to 0}{\longrightarrow} \varphi \quad s - L^p\big((0,1) \times Y^*\big).$$

Proof. First of all note that for any $x \in \mathbb{R}$ and $y_1 \in [0, L]$ we have

$$0 \leq x - \epsilon^\alpha \left[\frac{x}{\epsilon^\alpha}\right]_L L \leq \epsilon^\alpha L, \qquad \text{and} \qquad \epsilon^\alpha \left[\frac{x}{\epsilon^\alpha}\right]_L L + \epsilon^\alpha y_1 \stackrel{\epsilon\to 0}{\longrightarrow} x.$$

Therefore, for every $\varphi \in \mathcal{D}(0,1)$ with the definition of $\mathcal{T}_\epsilon(\cdot)$, see Definition 1.1.3, one gets

$$\begin{aligned} \|\mathcal{T}_\epsilon(\varphi) - \varphi\|^p_{L^p\left((0,1)\times Y^*\right)} &= \int_{I^\epsilon \times Y^*} |\mathcal{T}_\epsilon(\varphi) - \varphi|^p dx dy_1 dy_2 + \int_{\Lambda^\epsilon \times Y^*} |\varphi|^p dx dy_1 dy_2 \\ &\leq \int_{I^\epsilon \times Y^*} |m_\varphi(\epsilon^\alpha)|^p dx dy_1 dy_2 + \int_{\Lambda^\epsilon \times Y^*} |\varphi|^p dx dy_1 dy_2, \end{aligned}$$

where $m_\varphi(\epsilon^\alpha)$ is the modulus of continuity of the function φ

$$m_\varphi(\epsilon^\alpha) = \sup_{|x-y|<\epsilon^\alpha} |\varphi(x) - \varphi(y)|.$$

So, since φ is uniformly continuous on $[0,1]$ and the order of the measure of Λ^ϵ is ϵ^α we get the following strong convergence

$$\mathcal{T}_\epsilon(\varphi) \stackrel{\epsilon\to 0}{\longrightarrow} \varphi, \qquad \forall \varphi \in \mathcal{D}(0,1). \tag{1.1.4}$$

Finally, by density we prove the general case. If $\varphi \in L^p(0,1)$, $1 \leq p < \infty$, then there exist functions $\varphi_k \in \mathcal{D}(0,1)$ such that $\varphi_k \stackrel{\epsilon\to 0}{\longrightarrow} \varphi$ strongly in $L^p(0,1)$. Thus, we have

$$\begin{aligned} \|\mathcal{T}_\epsilon(\varphi) - \varphi\|_{L^p\left((0,1)\times Y^*\right)} &\leqslant \|\mathcal{T}_\epsilon(\varphi) - \mathcal{T}_\epsilon(\varphi_k)\|_{L^p\left((0,1)\times Y^*\right)} \\ +\|\mathcal{T}_\epsilon(\varphi_k) - \varphi_k\|_{L^p\left((0,1)\times Y^*\right)} &+ \|\varphi_k - \varphi\|_{L^p\left((0,1)\times Y^*\right)} \end{aligned}$$

from which the result is straightforward taking into account the convergence (1.1.4) and property vi) in Proposition 1.1.4. □

Proposition 1.1.11. *Let $\{\varphi^\epsilon\}$ be a sequence of functions in $L^p(0,1)$, $1 \leq p < \infty$, such that*

$$\varphi^\epsilon \stackrel{\epsilon\to 0}{\longrightarrow} \varphi \quad \textit{strongly in } L^p(0,1).$$

Then,

$$\mathcal{T}_\epsilon(\varphi^\epsilon) \stackrel{\epsilon\to 0}{\longrightarrow} \varphi \quad \textit{strongly in } L^p\big((0,1)\times Y^*\big).$$

Proof. By Minkowski's inequality we get

$$\|\mathcal{T}_\epsilon(\varphi^\epsilon) - \varphi\|_{L^p\left((0,1)\times Y^*\right)} \leq \|\mathcal{T}_\epsilon(\varphi^\epsilon) - \mathcal{T}_\epsilon(\varphi)\|_{L^p\left((0,1)\times Y^*\right)} + \|\mathcal{T}_\epsilon(\varphi) - \varphi\|_{L^p\left((0,1)\times Y^*\right)}.$$

On one hand by property vi) of Proposition 1.1.4 we get

$$\|\mathcal{T}_\epsilon(\varphi^\epsilon) - \mathcal{T}_\epsilon(\varphi)\|_{L^p\left((0,1)\times Y^*\right)} \leq C|||\varphi^\epsilon - \varphi|||_{L^p(R^\epsilon)},$$

with C independent of ϵ. Moreover, from hypothesis we have

$$\begin{aligned} |||\varphi^\epsilon - \varphi|||^p_{L^p(R^\epsilon)} &= \epsilon^{-1} \int_0^1 \int_{\epsilon b}^{\epsilon g(\frac{x}{\epsilon^\alpha})} |\varphi^\epsilon - \varphi|^p \, dx dy \\ &= \int_0^1 \Big(g\Big(\frac{x}{\epsilon^\alpha}\Big) - b\Big) |\varphi^\epsilon - \varphi|^p \, dx \\ &\leq (g_1 - b) \|\varphi^\epsilon - \varphi\|^p_{L^p(0,1)} \overset{\epsilon \to 0}{\longrightarrow} 0. \end{aligned}$$

Hence, $\|\mathcal{T}_\epsilon(\varphi^\epsilon) - \mathcal{T}_\epsilon(\varphi)\|_{L^p\big((0,1)\times Y^*\big)} \overset{\epsilon \to 0}{\longrightarrow} 0.$

On the other hand, by Proposition 1.1.10

$$\|\mathcal{T}_\epsilon(\varphi) - \varphi\|_{L^p\big((0,1)\times Y^*\big)} \overset{\epsilon \to 0}{\longrightarrow} 0.$$

Therefore, we have proved the desired convergence. □

Now, we get an important convergence result. For this, we will need to assume that the thin set R^ϵ is connected, b is strictly less than g_0, see hypothesis **(Hg)**. For simplicity, we assume that $b = 0$. In fact, the results that we prove below, Lemma 1.1.13 and Proposition 1.1.14, hold true for any positive b such that $0 < b < g_0$ with minor modifications. Then, we consider

$$R^\epsilon = \Big\{ (x, y) \in \mathbb{R}^2 \mid x \in (0,1), \ 0 < y < \epsilon \, g(x/\epsilon^\alpha) \Big\}. \tag{1.1.5}$$

where $g_0 = \min_{x \in \mathbb{R}} \{g(x)\} > 0$. Then, the sets R^ϵ and Y^* are connected domains.

Observe that, taking y_1 as a parameter, from the one-dimensional Poincaré-Wirtinger inequality we get

$$\Big\| \varphi - \frac{1}{g(y_1)} \int_0^{g(y_1)} \varphi \, dy_2 \Big\|^p_{L^p(0, g(y_1))} \leq C_p \int_0^{g(y_1)} \Big| \frac{\partial \varphi}{\partial y_2} \Big|^p dy_2, \ \forall \varphi \in W^{1,p}(Y^*).$$

Since $0 < g_0 \leq g(y_1) \leq g_1$ for all $y_1 \in \mathbb{R}$ we can ensure that C_p is uniformly bounded respect to y_1.

Furthermore we show in Lemma 1.1.13 that if the reference cell Y^* is connected, that is $0 < g_0$, then the Poincaré-Wirtinger inequality holds in Y^*. Recall that this inequality is defined as

Definition 1.1.12. *A bounded open set U satisfies the Poincaré-Wirtinger inequality for the exponents $1 \leq p < \infty$ if there exists a constant C_p such that*

$$\Big\| \varphi - \frac{1}{|U|} \int_U \varphi \, dx dy \Big\|_{L^p(U)} \leq C_p \|\nabla \varphi\|_{L^p(U)}, \quad \forall \varphi \in W^{1,p}(U).$$

Lemma 1.1.13. *Assume Y^* is connected, that is, $0 < g_0$. Then, the Poincaré-Wirtinger inequality for the exponents $1 < p < \infty$ holds in Y^*.*

Proof. Assume that the statement is not true and we will reach a contradiction. If the Poincaré-Wirtinger inequality does not hold in Y^* there exist a sequence $\{u_n\} \subset W^{1,p}(Y^*)$ with $||u_n - \mathcal{M}_{Y^*}(u_n)||_{L^p(Y^*)} \neq 0$ and such that

$$\frac{\displaystyle\int_{Y^*} |\nabla u_n|^p \, dy_2 dy_1}{\displaystyle\int_{Y^*} |u_n - \mathcal{M}_{Y^*}(u_n)|^p \, dy_2 dy_1} \to 0, \qquad \text{as } n \to 0. \tag{1.1.6}$$

Then, we define

$$w_n = \frac{u_n - \mathcal{M}_{Y^*}(u_n)}{||u_n - \mathcal{M}_{Y^*}(u_n)||_{L^p(Y^*)}}.$$

Notice that $||w_n||_{L^p(Y^*)} = 1$, $\mathcal{M}_{Y^*}(w_n) = 0$ and and taking into account (1.1.6) we have

$$\int_{Y^*} |\nabla w_n|^p \, dy_1 dy_2 \to 0, \qquad \text{as } n \to 0. \tag{1.1.7}$$

Now, we define the domain Y_0^* given by

$$Y_0^* = \Big\{(y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < L, 0 < y_2 < \frac{g_0}{2}\Big\} \subset Y^*.$$

Then, taking into account the properties of w_n we have

$$||w_n||_{L^p(Y_0^*)} \leq 1 \quad \text{ and } \quad \int_{Y_0^*} |\nabla w_n|^p \, dy_1 dy_2 \to 0, \qquad \text{as } n \to 0.$$

Thus, by weak compactness there exist $w_0 \in W^{1,p}(Y_0^*)$ such that, up to subsequences,

$$w_n \rightharpoonup w_0 \quad \text{w} - W^{1,p}(Y_0^*), \; s - L^p(Y_0^*)$$

Moreover since $||\nabla w_n||_{L^p(Y_0^*)} \to 0$ we have that w_0 is constant and the following convergence

$$w_n \to w_0 \quad \text{s} - W^{1,p}(Y_0^*).$$

Consider now the sequence $v_n = w_n - w_0 \in W^{1,p}(Y^*)$. Note that since $\nabla v_n = \nabla w_n$ we have

$$||\nabla v_n||_{L^p(Y^*)} \to 0. \tag{1.1.8}$$

Now we prove that

$$||v_n||_{L^p(Y^*)} \to 0.$$

Taking into account the fact that

$$v_n(y_1, y_2) - v_n(y_1, 0) = \int_0^{y_2} \frac{\partial v_n}{\partial y_2}(y_1, s) \, ds,$$

we have

$$||v_n||_{L^p(Y^*)} \leq C||v_n(\cdot, 0)||_{L^p(0,L)} + C\Big|\Big|\frac{\partial v_n}{\partial y_2}\Big|\Big|_{L^p(Y^*)}.$$

Therefore, using that the trace $v_n(\cdot, 0)$ goes strongly towards 0 in $L^p(0, L)$ since v_n converges to 0 strongly in $W^{1,p}(Y_0^*)$ and the convergence (1.1.8) we get

$$||w_n - w_0||_{L^p(Y^*)} = ||v_n||_{L^p(Y^*)} \to 0.$$

This convergence and (1.1.7) lead to

$$w_n \to w_0 \quad \mathrm{s} - W^{1,p}(Y^*).$$

Then, since by definition $\displaystyle\int_{Y^*} w_n \, dy_1 dy_2 = 0$ we get $w_0 = 0$ and therefore $\|w_n\|_{L^p(Y^*)} \to 0$. This is in contradiction with the definition of w_n for which $\|w_n\|_{L^p(Y^*)} = 1$.

$\square$

In the following Proposition we obtain interesting convergence results which do not depend on the value of the parameter α. To do that, we introduce a suitable decomposition of the functions $\varphi \in W^{1,p}(R^\epsilon)$. Actually, we write $\varphi(x, y) = V(x) + \varphi_r(x, y)$ where V is defined as follows

$$V(x) \equiv \frac{1}{\epsilon(g_0)} \int_0^{\epsilon g_0} \varphi(x, s) \, ds, \qquad \text{for a.e. } x \in (0, 1). \tag{1.1.9}$$

and $\varphi_r(x, y) \equiv \varphi(x, y) - V(x)$.

Proposition 1.1.14. *Let φ^ϵ be in $W^{1,p}(R^\epsilon)$ with $|||\varphi^\epsilon|||_{W^{1,p}(R^\epsilon)}$ uniformly bounded for some $1 < p < \infty$. Assume that R^ϵ is given by (1.1.5) and V^ϵ is defined as in (1.1.9). Then, there exists a function φ in $W^{1,p}(0, 1)$ such that, up to subsequences*

$$\begin{aligned}
&V^\epsilon \overset{\epsilon\to 0}{\rightharpoonup} \varphi \quad w - W^{1,p}(0, 1), \; s - L^p(0, 1) \\
&\mathcal{T}_\epsilon(V^\epsilon) \overset{\epsilon\to 0}{\rightharpoonup} \varphi \quad s - L^p\big((0, 1) \times Y^*\big), \\
&|||\varphi^\epsilon - V^\epsilon|||_{L^p(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0, \\
&|||\varphi^\epsilon - \varphi|||_{L^p(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0, \\
&\mathcal{T}_\epsilon(\varphi^\epsilon) \overset{\epsilon\to 0}{\rightharpoonup} \varphi \quad w - L^p\big((0, 1); W^{1,p}(Y^*)\big), \\
&\mathcal{T}_\epsilon(\varphi^\epsilon) \overset{\epsilon\to 0}{\longrightarrow} \varphi \;\; s - L^p\big((0, 1) \times Y^*\big).
\end{aligned}$$

Furthermore, there exists $\bar{\varphi} \in L^p((0, 1) \times Y^*)$ with $\frac{\partial \bar{\varphi}}{\partial y_2} \in L^p((0, 1) \times Y^*)$ such that, up to subsequences

$$\begin{aligned}
&\frac{1}{\epsilon}\mathcal{T}_\epsilon(\varphi_r^\epsilon) \overset{\epsilon\to 0}{\rightharpoonup} \bar{\varphi} \quad w\text{-}L^p((0, 1) \times Y^*), \\
&\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial y}\Big) \overset{\epsilon\to 0}{\rightharpoonup} \frac{\partial \bar{\varphi}}{\partial y_2} \quad w\text{-}L^p((0, 1) \times Y^*),
\end{aligned} \tag{1.1.10}$$

where $\varphi_r^\epsilon \equiv \varphi^\epsilon - V^\epsilon$.

Proof. First, we obtain estimates for the function V^ϵ. Note that since $\varphi^\epsilon \in W^{1,p}(R^\epsilon)$ and V^ϵ is given by

$$V^\epsilon(x) = \frac{1}{\epsilon g_0} \int_0^{\epsilon g_0} \varphi^\epsilon(x,s)\, ds.$$

we have $V^\epsilon \in W^{1,p}(0,1)$. Using Holder's inequality we get

$$\begin{aligned}
\|V^\epsilon\|_{L^p(0,1)} &= \Big(\int_0^1 \Big| \frac{1}{\epsilon g_0} \int_0^{\epsilon g_0} \varphi^\epsilon(x,s)\, ds \Big|^p dx \Big)^{\frac{1}{p}} \\
&\le \Big(\int_0^1 \frac{1}{\epsilon g_0} \int_0^{\epsilon g_0} |\varphi^\epsilon(x,s)|^p\, ds\, dx \Big)^{\frac{1}{p}} \\
&\le C\epsilon^{-\frac{1}{p}} \|\varphi^\epsilon\|_{L^p(R^\epsilon)} = C|||\varphi^\epsilon|||_{L^p(R^\epsilon)}, \\
\Big\| \frac{\partial V^\epsilon}{\partial x} \Big\|_{L^p(0,1)} &= \Big(\int_0^1 \Big| \frac{1}{\epsilon g_0} \int_0^{\epsilon g_0} \frac{\partial \varphi^\epsilon}{\partial x}(x,s)\, ds \Big|^p dx \Big)^{\frac{1}{p}} \\
&\le \Big(\int_0^1 \frac{1}{\epsilon g_0} \int_0^{\epsilon g_0} \Big| \frac{\partial \varphi^\epsilon}{\partial x}(x,s) \Big|^p ds\, dx \Big)^{\frac{1}{p}} \\
&\le C\epsilon^{-\frac{1}{p}} \Big\| \frac{\partial \varphi^\epsilon}{\partial x} \Big\|_{L^p(R^\epsilon)} = C\Big|\Big|\Big| \frac{\partial \varphi^\epsilon}{\partial x} \Big|\Big|\Big|_{L^p(R^\epsilon)}.
\end{aligned}$$

From these inequalities and taking into account that the norm $|||\varphi^\epsilon|||_{W^{1,p}(R^\epsilon)}$ is uniformly bounded we can ensure that there exists a function $\varphi \in W^{1,p}(0,1)$ such that, up to subsequences

$$V^\epsilon \overset{\epsilon \to 0}{\rightharpoonup} \varphi \quad \mathrm{w} - W^{1,p}(0,1) \quad \text{and } \mathrm{s} - L^p(0,1). \tag{1.1.11}$$

Consequently, it follows from Proposition 1.1.11 the following convergence as ϵ tends to zero

$$\mathcal{T}_\epsilon(V^\epsilon) \overset{\epsilon \to 0}{\longrightarrow} \varphi \quad \mathrm{s} - L^p\big((0,1) \times Y^*\big).$$

Recall that $\varphi_r^\epsilon \equiv \varphi^\epsilon - V^\epsilon$. Then, from the one-dimensional Poincaré-Wirtinger inequality and using a simple change of variables we get

$$\begin{aligned}
\int_0^{\epsilon g(x/\epsilon^\alpha)} |\varphi_r^\epsilon|^p dy &= \int_0^{\epsilon g(x/\epsilon^\alpha)} |\varphi^\epsilon - V^\epsilon|^p dy \\
&= \int_0^{\epsilon g(x/\epsilon^\alpha)} |\varphi^\epsilon(x,y) - \frac{1}{\epsilon g_0} \int_0^{\epsilon g_0} \varphi^\epsilon(x,s)\, ds|^p dy \\
&= \epsilon \int_0^{g(x/\epsilon^\alpha)} |\varphi^\epsilon(x,\epsilon x_2) - \frac{1}{g_0} \int_0^{g_0} \varphi^\epsilon(x,\epsilon t)\, dt|^p dx_2 \\
&\le C\epsilon \int_0^{g(x/\epsilon^\alpha)} \Big| \epsilon \frac{\partial \varphi^\epsilon}{\partial y}(x,\epsilon x_2) \Big|^p dx_2 \\
&= C\epsilon^p \int_0^{\epsilon g(x/\epsilon^\alpha)} \Big| \frac{\partial \varphi^\epsilon}{\partial y}(x,y) \Big|^p dy
\end{aligned}$$

Then, integrating in the interval $(0,1)$ we have

$$\int_{R^\epsilon} |\varphi_r^\epsilon|^p dy \le C\epsilon^p \int_{R^\epsilon} \Big| \frac{\partial \varphi^\epsilon}{\partial y}(x,y) \Big|^p dy,$$

which implies

$$|||\varphi_r^\epsilon|||_{L^p(R^\epsilon)} = |||\varphi^\epsilon - V^\epsilon|||_{L^p(R^\epsilon)} \le \epsilon \Big|\Big|\Big| \frac{\partial \varphi^\epsilon}{\partial x} \Big|\Big|\Big|_{L^p(R^\epsilon)}. \tag{1.1.12}$$

Therefore, since $|||\varphi^\epsilon|||_{W^{1,p}(R^\epsilon)}$ is uniformly bounded we obtatin

$$|||\varphi^\epsilon - V^\epsilon|||_{L^p(R^\epsilon)} \xrightarrow{\epsilon\to 0} 0.$$

Moreover, taking into account

$$|||\varphi^\epsilon - \varphi|||_{L^p(R^\epsilon)} \le |||\varphi^\epsilon - V^\epsilon|||_{L^p(R^\epsilon)} + |||V^\epsilon - \varphi|||_{L^p(R^\epsilon)},$$

and usingt (1.1.11) we immediately get

$$|||\varphi^\epsilon - \varphi|||_{L^p(R^\epsilon)} \xrightarrow{\epsilon\to 0} 0.$$

Hence, from property vi) in Proposition 1.1.4 we have

$$||\mathcal{T}_\epsilon(\varphi^\epsilon) - \mathcal{T}_\epsilon(\varphi)||_{L^p((0,1)\times Y^*)} \xrightarrow{\epsilon\to 0} 0.$$

Consequently, since $\mathcal{T}_\epsilon(\varphi) \xrightarrow{\epsilon\to 0} \varphi \quad \text{s} - L^p\big((0,1)\times Y^*\big)$, see Proposition 1.1.10, we obtain

$$\mathcal{T}_\epsilon(\varphi^\epsilon) \xrightarrow{\epsilon\to 0} \varphi \;\; \text{s} - L^p\big((0,1)\times Y^*\big).$$

Furthermore, since we have the following inequalities, they follow straightforward using properties vi) and viii) of Proposition 1.1.4 and $|||\varphi^\epsilon|||_{W^{1,p}(R^\epsilon)} \le C$,

$$\Big\|\frac{\partial}{\partial y_1}\mathcal{T}_\epsilon(\varphi^\epsilon)\Big\|_{L^p\big((0,1)\times Y^*\big)} \le \epsilon^\alpha C,$$
$$\Big\|\frac{\partial}{\partial y_2}\mathcal{T}_\epsilon(\varphi^\epsilon)\Big\|_{L^p\big((0,1)\times Y^*\big)} \le \epsilon C,$$

we obtain

$$\mathcal{T}_\epsilon(\varphi^\epsilon) \overset{\epsilon\to 0}{\rightharpoonup} \varphi \quad \text{w} - L^p\big((0,1); W^{1,p}(Y^*)\big).$$

Finally, we obtain convergences (1.1.10).

Observe that $\dfrac{\partial \varphi_r^\epsilon}{\partial y} = \dfrac{\partial \varphi^\epsilon}{\partial y}$. Then, taking into account property vii) in Proposition 1.1.4 we have

$$\frac{1}{\epsilon}\frac{\partial \mathcal{T}_\epsilon(\varphi_r^\epsilon)}{\partial y_2} = \mathcal{T}_\epsilon\Big(\frac{\partial \varphi_r^\epsilon}{\partial y}\Big) = \mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial y}\Big).$$

Hence, by using property vi) of Proposition 1.1.4 we get

$$\Big|\Big|\frac{1}{\epsilon}\frac{\partial \mathcal{T}_\epsilon(\varphi_r^\epsilon)}{\partial y_2}\Big|\Big|_{L^p((0,1)\times Y^*)} \le \Big|\Big|\Big|\frac{\partial \varphi^\epsilon}{\partial y}\Big|\Big|\Big|_{L^p(R^\epsilon)}. \tag{1.1.13}$$

Moreover, estimate (1.1.12) implies

$$\frac{1}{\epsilon}|||\varphi_r^\epsilon|||_{L^p(R^\epsilon)} \le \Big|\Big|\Big|\frac{\partial \varphi^\epsilon}{\partial y}\Big|\Big|\Big|_{L^p(R^\epsilon)}. \tag{1.1.14}$$

Therefore, since $|||\varphi^\epsilon|||_{W^{1,p}(R^\epsilon)}$ is uniformly bounded and in view of estimates (1.1.13) and (1.1.14) we can ensure that there exists a function $\bar{\varphi} \in L^2((0,1) \times Y^*)$ with $\dfrac{\partial \bar{\varphi}}{\partial y_2} \in L^p((0,1) \times Y^*)$ such that, up to subsequences

$$\begin{aligned} &\frac{1}{\epsilon}\mathcal{T}_\epsilon(\varphi_r^\epsilon) \overset{\epsilon \to 0}{\rightharpoonup} \bar{\varphi} \quad \text{w-}L^p((0,1) \times Y^*), \\ &\frac{1}{\epsilon}\frac{\partial \mathcal{T}_\epsilon(\varphi_r^\epsilon)}{\partial y_2} \overset{\epsilon \to 0}{\rightharpoonup} \frac{\partial \bar{\varphi}}{\partial y_2} \quad \text{w-}L^p((0,1) \times Y^*). \end{aligned}$$

Finally, since $\frac{1}{\epsilon}\frac{\partial \mathcal{T}_\epsilon(\varphi_r^\epsilon)}{\partial y_2} = \mathcal{T}_\epsilon\big(\frac{\partial \varphi^\epsilon}{\partial y}\big)$ we get

$$\mathcal{T}_\epsilon\big(\frac{\partial \varphi^\epsilon}{\partial y}\big) \overset{\epsilon \to 0}{\rightharpoonup} \frac{\partial \bar{\varphi}}{\partial y_2} \quad \text{w-}L^p((0,1) \times Y^*),$$

which ends the proof.

□

Finally, we introduce the averaging operator $\mathcal{U}_\epsilon$ which is the formal adjoint of the unfolding operator. We may consider the averaging operator as the inverse of the unfolding operator $\mathcal{T}_\epsilon$. We will use it to obtain some strong convergences and corrector results. Note that, to define and to obtain the main properties of the averaging operator $\mathcal{U}_\epsilon$ we may consider a general thin open set as in (1.1.1). That is, we do not need to assume that R^ϵ is connected.

Definition 1.1.15. *Let φ be a function in $L^p((0,1) \times Y^*)$, $p \in [1, \infty]$, then we set*

$$\mathcal{U}_\epsilon(\varphi)(x,y) = \begin{cases} \dfrac{1}{L}\displaystyle\int_0^L \varphi\Big(\epsilon^\alpha \Big[\frac{x}{\epsilon^\alpha}\Big]_L L + \epsilon^\alpha y_1, \Big\{\frac{x}{\epsilon^\alpha}\Big\}_L, \frac{y}{\epsilon}\Big)\, dy_1, & \forall (x,y) \in R_0^\epsilon, \\ 0 & \forall (x,y) \in R_1^\epsilon. \end{cases}$$

The following proposition provides de main properties of $\mathcal{U}_\epsilon$.

Proposition 1.1.16. *The averaging operator satisfies the following properties.*

i) $\mathcal{U}_\epsilon$ is the formal adjoint of the unfolding operator $\mathcal{T}_\epsilon$, in the sense that

$$\frac{1}{L}\int_{(0,1)\times Y^*} \mathcal{T}_\epsilon(\varphi)\psi \, dx dy_1 dy_2 = \frac{1}{\epsilon}\int_{R^\epsilon} \varphi \mathcal{U}_\epsilon(\psi)\, dx dy,$$

for $\varphi \in L^q(R^\epsilon)$ and $\psi \in L^p((0,1) \times Y^*)$ with $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

ii) The averaging operator $\mathcal{U}_\epsilon$ is linear and continuous from $L^p((0,1) \times Y^*)$ to $L^p(R^\epsilon)$, $1 \le p \le \infty$, and for every $\varphi \in L^p((0,1) \times Y^*)$ with $p \in [1, \infty)$ one has

$$|||\mathcal{U}_\epsilon(\varphi)|||_{L^p(R^\epsilon)} \leqslant \Big(\frac{1}{L}\Big)^{1/p} \|\varphi\|_{L^p\big((0,1)\times Y^*\big)}.$$

iii) $\mathcal{U}_\epsilon$ *is "almost" the left inverse of* $\mathcal{T}_\epsilon$ *in the sense that for every* $\varphi \in L^p(R^\epsilon), 1 \leq p \leq \infty$*, we have*

$$\mathcal{U}_\epsilon\big(\mathcal{T}_\epsilon(\varphi)\big)(x,y) = \begin{cases} \varphi(x,y) & for \quad (x,y) \in R_0^\epsilon, \\ 0 & for \quad (x,y) \in R_1^\epsilon, \end{cases}$$

iv) Let $\phi \in L^p(0,1)$*,* $1 \leq p < \infty$*. Then,*

$$|||\mathcal{U}_\epsilon(\phi) - \phi|||_{L^p(R^\epsilon)} \xrightarrow{\epsilon\to 0} 0.$$

v) Let $\{\varphi^\epsilon\}$ *be a sequence in* $L^p(R^\epsilon)$*,* $p \in [1,\infty)$*, such that it satisfies*

$$\mathcal{T}_\epsilon(\varphi^\epsilon) \xrightarrow{\epsilon\to 0} \varphi \ s - L^p\big((0,1)\times Y^*\big) \quad and \quad \frac{1}{\epsilon}\int_{R_1^\epsilon} |\varphi^\epsilon|^p dxdy \xrightarrow{\epsilon\to 0} 0.$$

Then, $|||\mathcal{U}_\epsilon(\varphi) - \varphi^\epsilon|||_{L^p(R^\epsilon)} \xrightarrow{\epsilon\to 0} 0.$

Proof. i) Let φ be in $L^q(R^\epsilon)$ and let ψ be in $L^p((0,1)\times Y^*)$ we have

$$\begin{aligned}
&\frac{1}{L}\int_{(0,1)\times Y^*} \mathcal{T}_\epsilon(\varphi)\psi \, dxdy_1dy_2 \\
&= \frac{1}{L}\sum_{k=0}^{N_\epsilon}\int_{k\epsilon^\alpha L}^{(k+1)\epsilon^\alpha L}\int_{Y^*} \varphi\Big(\epsilon^\alpha\Big[\frac{x}{\epsilon^\alpha}\Big]_L L + \epsilon^\alpha y_1, \epsilon y_2\Big)\psi(x,y_1,y_2)\, dy_1dy_2dx \\
&= \frac{1}{L}\sum_{k=0}^{N_\epsilon}\int_0^L\int_{Y^*} \epsilon^\alpha\varphi\Big(\epsilon^\alpha kL + \epsilon^\alpha y_1, \epsilon y_2\Big)\psi(\epsilon^\alpha kL + \epsilon^\alpha z, y_1, y_2))\, dy_1dy_2dz \\
&= \frac{1}{L}\sum_{k=0}^{N_\epsilon}\int_0^L\int_{k\epsilon^\alpha L}^{(k+1)\epsilon^\alpha L}\int_{\epsilon b}^{\epsilon g(x/\epsilon^\alpha)} \frac{1}{\epsilon}\varphi(x,y)\psi\Big(\epsilon^\alpha kL + \epsilon^\alpha z, \Big\{\frac{x}{\epsilon^\alpha}\Big\}_L, \frac{y}{\epsilon}\Big)\, dydxdz \\
&= \frac{1}{\epsilon}\int_{R^\epsilon} \varphi\mathcal{U}_\epsilon(\psi)\, dxdy.
\end{aligned}$$

ii) The proof is obvious for $p = 1$ due to the duality above. For $p > 1$, using property i) of $\mathcal{U}_\epsilon$, Hölder's inequality and property vi) of Proposition 1.1.4 we get

$$\begin{aligned}
|||\mathcal{U}_\epsilon(\varphi)|||^p_{L^p(R^\epsilon)} &= \frac{1}{\epsilon}\int_{R^\epsilon} |\mathcal{U}_\epsilon(\varphi)^{p-1}\mathcal{U}_\epsilon(\varphi)|\, dxdy \\
&= \frac{1}{L}\int_{(0,1)\times Y^*} \big|\mathcal{T}_\epsilon\big(\mathcal{U}_\epsilon(\varphi)\big)^{p-1}\varphi\big|\, dxdy_1dy_2 \\
&\leq \frac{1}{L}||\mathcal{T}_\epsilon\big(\mathcal{U}_\epsilon(\varphi)\big)^{p-1}||_{L^{\frac{p}{p-1}}\big((0,1)\times Y^*\big)}||\varphi||_{L^p\big((0,1)\times Y^*\big)} \\
&\leq \Big(\frac{L}{\epsilon}\Big)^{\frac{p}{p-1}}\frac{1}{L}||\mathcal{U}_\epsilon(\varphi)^{p-1})||_{L^{\frac{p}{p-1}}(R^\epsilon)}||\varphi||_{L^p\big((0,1)\times Y^*\big)}
\end{aligned}$$

Therefore, since $\dfrac{p-1}{p} + \dfrac{1}{p} = 1$ we find through an easy computation the relation between the norms

$$|||\mathcal{U}_\epsilon(\varphi)|||_{L^p(R^\epsilon)} \leqslant \Big(\frac{1}{L}\Big)^{1/p} \|\varphi\|_{L^p\big((0,1)\times Y^*\big)}.$$

iii) It is immediate from the definition of the $\mathcal{U}_\epsilon$ and $\mathcal{T}_\epsilon$.

iv) We first prove the assertion for every $\phi \in \mathcal{D}(0,1)$.

$$|||\mathcal{U}_\epsilon(\phi) - \phi|||^p_{L^p(R^\epsilon)} = \frac{1}{\epsilon}\int_{R_0^\epsilon} \Big|\frac{1}{L}\int_0^L \phi\Big(\epsilon^\alpha \Big[\frac{x}{\epsilon^\alpha}\Big]_L L + \epsilon^\alpha y_1\Big)\, dy_1 - \phi(x)\Big|^p dxdy$$
$$+ \frac{1}{\epsilon}\int_{R_1^\epsilon} |\phi(x)|^p\, dxdy \le \frac{1}{\epsilon}\int_{R_0^\epsilon} |m_\phi(\epsilon^\alpha)|^p\, dxdy + \frac{1}{\epsilon}\int_{R_1^\epsilon} |\phi(x)|^p\, dxdy$$

where $m_\phi(\epsilon^\alpha)$ is the modulus of continuity of the function ϕ. Then, since ϕ is uniformly continuous in $[0,1]$ and $|\phi(x)|^p$ verifies the unfolding criterion for integrals, see Proposition 1.1.7 we have the desired convergence

$$|||\mathcal{U}_\epsilon(\phi) - \phi|||_{L^p(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0.$$

If $\phi \in L^p(0,1)$, let $\phi_k \in \mathcal{D}(0,1)$ such that $\phi_k \overset{\epsilon\to 0}{\longrightarrow} \phi$ strongly in $L^p(0,1)$. Then, taking into account ii) we have

$$\begin{aligned}
|||\mathcal{U}_\epsilon(\phi) - \phi|||_{L^p(R^\epsilon)} \le & |||\mathcal{U}_\epsilon(\phi) - \mathcal{U}_\epsilon(\phi_k)|||_{L^p(R^\epsilon)} \\
& + |||\mathcal{U}_\epsilon(\phi_k) - \phi_k|||_{L^p(R^\epsilon)} + |||\phi_k - \phi|||_{L^p(R^\epsilon)} \\
\le & \Big(\frac{1}{L}\Big)^{-1/p} \|\phi - \phi_k\|_{L^p\big((0,1)\times Y^*\big)} \\
& + |||\mathcal{U}_\epsilon(\phi_k) - \phi_k|||_{L^p(R^\epsilon)} + |||\phi_k - \phi|||_{L^p(R^\epsilon)}.
\end{aligned}$$

From this inequality the convergence is straightforward.

v) Using properties ii) and iii) of $\mathcal{U}_\epsilon$ we obtain

$$\begin{aligned}
|||\mathcal{U}_\epsilon(\varphi) - \varphi^\epsilon|||^p_{L^p(R^\epsilon)} = & \frac{1}{\epsilon}\int_{R_0^\epsilon} \Big|\mathcal{U}_\epsilon\Big(\varphi - \mathcal{T}_\epsilon(\varphi^\epsilon)\Big)\Big|^p dxdy + \frac{1}{\epsilon}\int_{R_1^\epsilon} |\varphi^\epsilon|^p\, dxdy \\
\le & \frac{1}{L}\|\varphi - \mathcal{T}_\epsilon(\varphi^\epsilon)\|^p_{L^p\big((0,1)\times Y^*\big)} + \frac{1}{\epsilon}\int_{R_1^\epsilon} |\varphi^\epsilon|^p\, dxdy,
\end{aligned}$$

which goes to zero.

□

1.2. The resonant case, $\alpha = 1$

In this section we apply the periodic unfolding operator introduced in the previous section in order to obtain the limit problem of (1.0.2) when the parameter α is equal to 1.

We start describing the problem for this particular case. We consider a two-dimensional thin domain given by

$$R^\epsilon = \Big\{(x, y) \in \mathbb{R}^2 \mid x \in (0, 1),\ 0 < y < \epsilon\, g(x/\epsilon)\Big\}, \tag{1.2.1}$$

where $g(\cdot)$ satisfies the hypothesis **($\mathbf{H_g}$)** stated at the beginning of Section 1.1.

In addition, we require that $0 < g_0$ which in particular implies that the Poincaré-Wirtinger inequality holds, see Definition 1.1.12, for the exponent $p \in (1, \infty)$ in the representative cell

$$Y^* = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \in (0, L),\ 0 < y_2 < g(y_1)\}.$$

Remark 1.2.1. *Observe that since $0 < g_0$, R^ϵ and Y^* are connected.*

Remark 1.2.2. *Note that, as we have mentioned in the Introduction these assumptions are weaker than the hypothesis assumed on the boundary of R^ϵ necessary for the existence of extension operators. Therefore, we may consider a larger class of thin domains than in previous works, see Figure 1.2.*

Notice that the domain R^ϵ shrinks in the vertical direction and it has an oscillatory behavior at the top boundary given by the function g. We say that this problem is resonant because the order of compression of the thin domain, the amplitude and period of the oscillations of the top boundary are of the same order ϵ.

First we state a compactness result which allows us to identify the limit of the image of the gradient of a uniformly bounded sequence by the unfolding operator method. Observe that as far as the homogenization theory is concerned this is an essential result because it gives us the relation between the limit of the solution and the limit of its gradient which is one of the main difficulties when passing to the limit in homogenized type problems.

Theorem 1.2.3. *Let $\varphi^\epsilon \in W^{1,p}(R^\epsilon)$ for every $\epsilon > 0$ and $|||\varphi^\epsilon|||_{W^{1,p}(R^\epsilon)}$ uniformly bounded for some $1 < p < \infty$. Then, there exist functions $\varphi \in W^{1,p}(0,1)$ and $\varphi_1 \in L^p\big((0,1); W^{1,p}_{\#}(Y^*)\big)$ such that, up to subsequences*

i) $\mathcal{T}_\epsilon(\varphi^\epsilon) \overset{\epsilon \to 0}{\rightharpoonup} \varphi \quad w - L^p\big((0,1); W^{1,p}(Y^*)\big)$,

ii) $\mathcal{T}_\epsilon\Big(\dfrac{\partial \varphi^\epsilon}{\partial x}\Big) \overset{\epsilon \to 0}{\rightharpoonup} \dfrac{\partial \varphi}{\partial x} + \dfrac{\partial \varphi_1}{\partial y_1} \quad w - L^p\big((0,1) \times Y^*\big)$,

$\mathcal{T}_\epsilon\Big(\dfrac{\partial \varphi^\epsilon}{\partial y}\Big) \overset{\epsilon \to 0}{\rightharpoonup} \dfrac{\partial \varphi_1}{\partial y_2} \quad w - L^p\big((0,1) \times Y^*\big)$.

Proof.
i) This convergence was obtained in Proposition 1.1.14 for any α greater than 0.

ii) Following similar arguments as in [46] we introduce the function Z_ϵ defined in $(0,1) \times Y^*$ as follows

$$Z_\epsilon(x, y_1, y_2) \equiv \frac{1}{\epsilon}\Big(\mathcal{T}_\epsilon(\varphi^\epsilon)(x, y_1, y_2) - \frac{1}{|Y^*|}\int_{Y^*} T_\epsilon(\varphi^\epsilon)(x, y_1, y_2)\, dy_1 dy_2\Big).$$

Observe that $Z_\epsilon(x,\cdot,\cdot)$ has mean value zero in Y^*. Moreover, it satisfies

$$\frac{\partial Z_\epsilon}{\partial y_1} = \frac{1}{\epsilon}\frac{\partial}{\partial y_1}\mathcal{T}_\epsilon(\varphi^\epsilon) = \mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial x}\Big), \quad \frac{\partial Z_\epsilon}{\partial y_2} = \frac{1}{\epsilon}\frac{\partial}{\partial y_2}\mathcal{T}_\epsilon(\varphi^\epsilon) = \mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial y}\Big).$$

where we have used property vii) of Proposition 1.1.4.
Hence, in order to get ii) we will prove that

$$Z_\epsilon \overset{\epsilon\to 0}{\rightharpoonup} \varphi_1 + y_1^c \frac{\partial \varphi}{\partial x} \quad \mathrm{w} - L^p\big((0,1); W^{1,p}(Y^*)\big), \tag{1.2.2}$$

where $s^c \equiv s - \dfrac{1}{|Y^*|}\displaystyle\int_{Y^*} y_1 \, dy_1 dy_2$, for any $s \in \mathbb{R}$.
Observe that by Proposition 1.1.4, viii), and using that $|||\varphi^\epsilon|||_{W^{1,p}(R^\epsilon)} \le C$ for some constant C independent of ϵ, the sequences $\left\{\frac{\partial Z_\epsilon}{\partial y_1}\right\}$ and $\left\{\frac{\partial Z_\epsilon}{\partial y_2}\right\}$ are bounded in $L^p\big((0,1)\times Y^*\big)$. Then, applying the Poincaré-Wirtinger inequality to the function $(y_1,y_2) \to Z_\epsilon(x,y_1,y_2) - y_1^c \frac{\partial \varphi}{\partial x}(x)$ (which is also of mean value zero in Y^*) we obtain

$$\Big|\Big| Z_\epsilon - y_1^c \frac{\partial \varphi}{\partial x}\Big|\Big|_{L^p((0,1)\times Y^*)} \leqslant C.$$

Hence, we can conclude that there is a function $\varphi_1 \in L^p\big((0,1); W^{1,p}(Y^*)\big)$ such that, up to subsequences

$$Z_\epsilon - y_1^c \frac{\partial \varphi}{\partial x} \overset{\epsilon\to 0}{\rightharpoonup} \varphi_1 \quad \mathrm{w} - L^p\big((0,1); W^{1,p}(Y^*)\big),$$

which in particular it implies that

$$\begin{aligned}
\frac{\partial}{\partial y_1}\Big(Z_\epsilon - y_1^c \frac{\partial \varphi}{\partial x}\Big) &= \mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial x}\Big) - \frac{\partial \varphi}{\partial x} \overset{\epsilon\to 0}{\rightharpoonup} \frac{\partial \varphi_1}{\partial y_1} \quad \mathrm{w} - L^p\big((0,1)\times Y^*\big),\\
\frac{\partial}{\partial y_2}\Big(Z_\epsilon - y_1^c \frac{\partial \varphi}{\partial x}\Big) &= \mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial y}\Big) \overset{\epsilon\to 0}{\rightharpoonup} \frac{\partial \varphi_1}{\partial y_2} \quad \mathrm{w} - L^p\big((0,1)\times Y^*\big).
\end{aligned}$$

Equivalently,

$$\begin{aligned}
&\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial x}\Big) \overset{\epsilon\to 0}{\rightharpoonup} \frac{\partial \varphi}{\partial x}(x) + \frac{\partial \varphi_1}{\partial y_1}(x,y_1,y_2) \quad \mathrm{w} - L^p\big((0,1)\times Y^*\big),\\
&\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial y}\Big) \overset{\epsilon\to 0}{\rightharpoonup} \frac{\partial \varphi_1}{\partial y_2}(x,y_1,y_2) \quad \mathrm{w} - L^p\big((0,1)\times Y^*\big).
\end{aligned}$$

To conclude the arguments we just need to show the $L-$periodicity of the function φ_1. On one hand, let $\psi \in \mathcal{D}\big((0,1)\times Y^*\big)$, using an obvious change of variable we have

$$\int_{(0,1)\times Y^*} [Z_\epsilon(x,y_1+L,y_2) - Z_\epsilon(x,y_1,y_2)]\psi(x,y_1,y_2) dx dy_1 dy_2$$

$$
\begin{aligned}
&= \int_{(0,1)\times Y^*} \frac{1}{\epsilon}\varphi^\epsilon\Big(\epsilon\Big[\frac{x}{\epsilon}\Big]_L L + \epsilon(y_1+L), \epsilon y_2\Big)\psi(x,y_1,y_2)\,dxdy_1dy_2 \\
&- \int_{(0,1)\times Y^*} \frac{1}{\epsilon}\varphi^\epsilon\Big(\epsilon\Big[\frac{x}{\epsilon}\Big]_L L + \epsilon y_1, \epsilon y_2\Big)\psi(x,y_1,y_2)\,dxdy_1dy_2 \\
&= \int_{(0,1)\times Y^*} \varphi^\epsilon\Big(\epsilon\Big[\frac{x}{\epsilon}\Big]_L L + \epsilon y_1, \epsilon y_2\Big)[\frac{\psi(x-\epsilon L,y_1,y_2)-\psi(x,y_1,y_2)}{\epsilon}]dxdy_1dy_2.
\end{aligned}
$$

where we have used that we can consider ϵ small enough such that the Hausdorff distance between the support of the function ψ and $\partial((0,1)\times Y^*)$ is greater than ϵL.

Note that, by assertion i) the last integral converges to

$$
\int_{(0,1)\times Y^*} -L\varphi\frac{\partial\psi}{\partial x}dxdy_1dy_2.
$$

Hence,

$$
\begin{aligned}
&\int_{(0,1)\times Y^*} [Z_\epsilon(x,y_1+L,y_2) - Z_\epsilon(x,y_1,y_2)]\psi(x,y_1,y_2)dxdy_1dy_2 \\
&\qquad \stackrel{\epsilon\to 0}{\longrightarrow} \int_{(0,1)\times Y^*} -L\varphi\frac{\partial\psi}{\partial x}dxdy_1dy_2.
\end{aligned} \tag{1.2.3}
$$

On the other hand, by the definition of weak derivative we have

$$
\begin{aligned}
&\int_{(0,1)\times Y^*} \Big((y_1+L)^c - y_1^c\Big)\frac{\partial\varphi}{\partial x}\psi\; dxdy_1dy_2 \\
&= \int_{(0,1)\times Y^*} L\frac{\partial\varphi}{\partial x}\psi\; dxdy_1dy_2 = -\int_{(0,1)\times Y^*} L\varphi\frac{\partial\psi}{\partial x}dxdy_1dy_2.
\end{aligned} \tag{1.2.4}
$$

Then, in view of (1.2.3) and (1.2.4) we obtain

$$
\begin{aligned}
&\int_{(0,1)\times Y^*} \Big[\Big(Z_\epsilon(x,y_1+L,y_2) - (y_1+L)^c\frac{\partial\varphi}{\partial x}\Big)\Big]\psi(x,y_1,y_2)\; dxdy_1dy_2 \\
&- \int_{(0,1)\times Y^*} \Big[\Big(Z_\epsilon(x,y_1,y_2)\; - y_1^c\frac{\partial\varphi}{\partial x}\Big)\Big]\psi(x,y_1,y_2)\; dxdy_1dy_2 \stackrel{\epsilon\to 0}{\longrightarrow} 0.
\end{aligned} \tag{1.2.5}
$$

Hence, from convergence (1.2.2) and (1.2.5) we obtain for all $\psi \in \mathcal{D}\big((0,1)\times Y^*\big)$

$$
\int_{(0,1)\times Y^*} [\varphi_1(x,y_1+L,y_2) - \varphi_1(x,y_1,y_2)]\psi(x,y_1,y_2)\; dxdy_1dy_2 = 0,
$$

which implies that $\varphi_1(x,y_1+L,y_2) - \varphi_1(x,y_1,y_2)$ a.e. $(x,y_1,y_2) \in (0,1)\times Y^*$. Therefore, φ_1 is $L-$periodic in the variable y_1.

□

We can now state the homogenization theorem. We will see that the proof is now very straightforward using the previous compactness result.

Theorem 1.2.4. *Let u^ϵ be the solution of problem (1.0.3) with $f^\epsilon \in L^2(R^\epsilon)$ satisfying $|||f^\epsilon|||_{L^2(R^\epsilon)} \leq C$ for some positive constant C independent of the parameter $\epsilon > 0$. Assume that there exists $\hat{f} \in L^2((0,1) \times Y^*)$ such that*

$$\mathcal{T}_\epsilon(f^\epsilon) \overset{\epsilon \to 0}{\rightharpoonup} \hat{f} \text{ weakly in } L^2\big((0,1) \times Y^*\big).$$

Then, there exist $u \in H^1(0,1)$ and $u_1 \in L^2\Big((0,1); H^1_\#(Y^*)\Big)$ such that

$$\mathcal{T}_\epsilon(u^\epsilon) \overset{\epsilon \to 0}{\rightharpoonup} u \quad \text{weakly in } L^2\big((0,1); H^1(Y^*)\big),$$

$$\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big) \overset{\epsilon \to 0}{\rightharpoonup} \frac{\partial u}{\partial x}(x) + \frac{\partial u_1}{\partial y_1}(x, y_1, y_2) \quad \text{weakly in } L^2\big((0,1) \times Y^*\big), \tag{1.2.6}$$

$$\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial y}\Big) \overset{\epsilon \to 0}{\rightharpoonup} \frac{\partial u_1}{\partial y_2}(x, y_1, y_2) \qquad \text{weakly in } L^2\big((0,1) \times Y^*\big), \tag{1.2.7}$$

and the pair (u, u_1) is the unique solution in $H^1(0,1) \times L^2\Big((0,1); H^1_\#(Y^*)/\mathbb{R}\Big)$ of the problem

$$\begin{cases} \forall \phi \in H^1(0,1), \forall \varphi \in L^2\Big((0,1); H^1_\#\big(Y^*(x)\big)\Big) \\ \displaystyle\int_{(0,1)\times Y^*} \Big\{ \Big(\frac{\partial u}{\partial x}(x) + \frac{\partial u_1}{\partial y_1}(x, y_1, y_2) \Big) \Big(\frac{\partial \phi}{\partial x}(x) + \frac{\partial \varphi}{\partial y_1}(x, y_1, y_2) \Big) \Big\} dx dy_1 dy_2 \\ \qquad + \displaystyle\int_{(0,1)\times Y^*} \Big\{ \frac{\partial u_1}{\partial y_2}(x, y_1, y_2) \frac{\partial \varphi}{\partial y_2}(x, y_1, y_2) + u(x)\phi(x) \Big\} dx dy_1 dy_2 \\ \qquad\qquad = \displaystyle\int_{(0,1)\times Y^*} \hat{f}(x, y_1, y_2)\phi(x) dx dy_1 dy_2. \end{cases} \tag{1.2.8}$$

Moreover, $u \in H^1(0,1)$ is the unique weak solution of the following Neumann problem

$$\begin{cases} -q_0 u_{xx} + u = f_0(x), \quad x \in (0,1), \\ u'(0) = u'(1) = 0, \end{cases} \tag{1.2.9}$$

where $f_0 = \dfrac{1}{|Y^*|} \displaystyle\int_{Y^*} \hat{f} dy_1 dy_2$, the homogenized coefficient is defined by

$$q_0 = \frac{1}{|Y^*|} \int_{Y^*} \Big\{ 1 - \frac{\partial X}{\partial y_1}(y_1, y_2) \Big\} dy_1 dy_2,$$

and $X \in H^1_\#(Y^*)$ satisfying $\displaystyle\int_{Y^*} X \, dy_1 dy_2 = 0$ is the unique solution of the following problem

$$\int_{Y^*} \nabla X \nabla \psi \, dy_1 dy_2 = \int_{Y^*} \frac{\partial \psi}{\partial y_1} dy_1 dy_2, \quad \forall \psi \in H^1_\#(Y^*). \tag{1.2.10}$$

Proof. First of all, we obtain the a priori estimate satisfied by u^ϵ, solution of the variational problem (1.0.3). Considering u^ϵ as a test function in (1.0.3) we obtain

$$\Big\|\frac{\partial u^\epsilon}{\partial x}\Big\|^2_{L^2(R^\epsilon)} + \Big\|\frac{\partial u^\epsilon}{\partial y}\Big\|^2_{L^2(R^\epsilon)} + \|u^\epsilon\|^2_{L^2(R^\epsilon)} \leq \|f^\epsilon\|_{L^2(R^\epsilon)}\|u^\epsilon\|_{L^2(R^\epsilon)}.$$

Then, using that $|||f^\epsilon|||_{L^2(R^\epsilon)} \leq C$, with C independent of ϵ, we deduce that

$$|||u^\epsilon|||_{H^1(R^\epsilon)} \leq C.$$

Hence, in view of Theorem 1.2.3 follows that there exist $u \in H^1(0,1)$ and $u_1 \in L^2\big((0,1); H^1_\#(Y^*)\big)$ such that one has, at least for a subsequence, the following convergences

$$\begin{aligned}
&\mathcal{T}_\epsilon(u^\epsilon) \overset{\epsilon\to 0}{\rightharpoonup} u \quad \text{weakly in } \ L^2\big((0,1); H^1(Y^*)\big),\\
&\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big) \overset{\epsilon\to 0}{\rightharpoonup} \frac{\partial u}{\partial x}(x) + \frac{\partial u_1}{\partial y_1}(x,y_1,y_2) \quad \text{weakly in } \ L^2\big((0,1)\times Y^*\big),\\
&\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial y}\Big) \overset{\epsilon\to 0}{\rightharpoonup} \frac{\partial u_1}{\partial y_2}(x,y_1,y_2) \qquad \text{weakly in } \ L^2\big((0,1)\times Y^*\big).
\end{aligned} \tag{1.2.11}$$

Now we prove that u and u_1 satisfy (1.2.8). We apply the unfolding operator to the variational formulation (1.0.3), by property v) in Proposition 1.1.4, we have

$$\begin{aligned}
&\int_{(0,1)\times Y^*} \Big\{\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big)\mathcal{T}_\epsilon\Big(\frac{\partial \phi}{\partial x}\Big) + \mathcal{T}_\epsilon(u^\epsilon)\mathcal{T}_\epsilon(\phi)\Big\}\, dxdy_1dy_2\\
&\qquad + \frac{L}{\epsilon}\int_{R_1^\epsilon} \Big\{\frac{\partial u^\epsilon}{\partial x}\frac{\partial \phi}{\partial x} + u^\epsilon\phi\Big\}\, dxdy\\
&\qquad = \int_{(0,1)\times Y^*} \mathcal{T}_\epsilon(f^\epsilon)\mathcal{T}_\epsilon(\phi) dxdy_1dy_2 + \frac{L}{\epsilon}\int_{R_1^\epsilon} f^\epsilon\phi, \ \ \forall \phi \in H^1(0,1).
\end{aligned} \tag{1.2.12}$$

Note that, since R^ϵ degenerates into a line and the limit problem will be one dimensional, we have taken $\phi \in H^1(0,1)$ as a test function in (1.0.3) and then, the derivative respect to y does not appear.

Taking into account convergences (1.2.11), Proposition 1.1.8, Proposition 1.1.10 and the hypothesis on the function f^ϵ we may pass to the limit in (1.2.12) to obtain

$$\begin{aligned}
&\int_{(0,1)\times Y^*} \Big\{\Big(\frac{\partial u}{\partial x}(x) + \frac{\partial u_1}{\partial y_1}(x,y_1)\Big)\frac{\partial \phi}{\partial x}(x) + u(x)\phi(x)\Big\}\, dxdy_1dy_2\\
&\qquad = \int_{(0,1)\times Y^*} \hat{f}(x,y_1,y_2)\phi(x)\, dxdy_1dy_2 \quad \forall \phi \in H^1(0,1).
\end{aligned} \tag{1.2.13}$$

Observe that the terms in (1.2.12) with $\dfrac{1}{\epsilon}$ disappear in the limit from Proposition 1.1.8.

We choose now as a test function in (1.0.3) the function v^ϵ defined by

$$v^\epsilon(x,y) = \epsilon\phi(x)\psi\Big(\frac{x}{\epsilon}, \frac{y}{\epsilon}\Big), \quad (x,y) \in R^\epsilon.$$

where $\phi \in \mathcal{D}(0,1)$ and $\psi \in H^1_{\#}(Y^*)$. Observe that, in view of property iv) in Proposition 1.1.4, v^ϵ is well defined and it is obvious from its definition that it satisfies

$$\begin{aligned}
&v^\epsilon \in H^1(R^\epsilon), \quad \mathcal{T}_\epsilon(v^\epsilon) = \epsilon\mathcal{T}_\epsilon(\phi)\psi,\\
&\frac{\partial v^\epsilon}{\partial x} = \epsilon\frac{\partial \phi}{\partial x}\psi\Big(\frac{x}{\epsilon},\frac{y}{\epsilon}\Big) + \phi\frac{\partial \psi}{\partial y_1}\Big(\frac{x}{\epsilon},\frac{y}{\epsilon}\Big),\\
&\frac{\partial v^\epsilon}{\partial y} = \phi\frac{\partial \psi}{\partial y_1}\Big(\frac{x}{\epsilon},\frac{y}{\epsilon}\Big).
\end{aligned}$$

Hence, using the properties of the unfolding operator we get

$$\begin{aligned}
\mathcal{T}_\epsilon(v^\epsilon) &\overset{\epsilon\to 0}{\longrightarrow} 0 \quad \text{s-}L^2((0,1)\times Y^*),\\
\mathcal{T}_\epsilon\Big(\frac{\partial v^\epsilon}{\partial x}\Big) &\overset{\epsilon\to 0}{\longrightarrow} \phi\frac{\partial \psi}{\partial y_1} \quad \text{s-}L^2((0,1)\times Y^*),\\
\mathcal{T}_\epsilon\Big(\frac{\partial v^\epsilon}{\partial y}\Big) &\overset{\epsilon\to 0}{\longrightarrow} \phi\frac{\partial \psi}{\partial y_2} \quad \text{s-}L^2((0,1)\times Y^*).
\end{aligned}$$

Then, taking v^ϵ as test function in the weak formulation (1.0.3) and applying the unfolding operator we get

$$\begin{aligned}
&\int_{(0,1)\times Y^*}\Big\{\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big)\mathcal{T}_\epsilon\Big(\frac{\partial v^\epsilon}{\partial x}\Big) + \mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial y}\Big)\mathcal{T}_\epsilon\Big(\frac{\partial v^\epsilon}{\partial y}\Big) + \mathcal{T}_\epsilon(u^\epsilon)\mathcal{T}_\epsilon(v^\epsilon)\Big\}\,dxdy_1dy_2\\
&= \int_{(0,1)\times Y^*}\mathcal{T}_\epsilon(f^\epsilon)\mathcal{T}_\epsilon(v^\epsilon)dxdy_1dy_2 + \frac{L}{\epsilon}\int_{R_1^\epsilon}\Big\{f^\epsilon v^\epsilon - \frac{\partial u^\epsilon}{\partial x}\frac{\partial v^\epsilon}{\partial x} - \frac{\partial u^\epsilon}{\partial y}\frac{\partial v^\epsilon}{\partial y} - u^\epsilon v^\epsilon\Big\}\,dxdy,
\end{aligned}$$

which by Proposition 1.1.9 and the convergences above gives at the limit

$$\int_{(0,1)\times Y^*}\Big\{\Big(\frac{\partial u}{\partial x}+\frac{\partial u_1}{\partial y_1}\Big)\phi\frac{\partial \psi}{\partial y_1}+\frac{\partial u_1}{\partial y_2}\phi\frac{\partial \psi}{\partial y_2}\Big\}\,dxdy_1dy_2 = 0, \quad \forall\phi\in\mathcal{D}(0,1),\ \psi\in H^1_{\#}(Y^*).$$

By the density of the tensor product $\mathcal{D}(0,1)\otimes H^1_{\#}(Y^*)$ in $L^2\big((0,1);H^1_{\#}(Y^*)\big)$, the equality holds true

$$\int_{(0,1)\times Y^*}\Big\{\Big(\frac{\partial u}{\partial x}+\frac{\partial u_1}{\partial y_1}\Big)\frac{\partial \varphi}{\partial y_1}+\frac{\partial u_1}{\partial y_2}\frac{\partial \varphi}{\partial y_2}\Big\}\,dxdy_1dy_2 = 0,\ \forall\varphi\in L^2\big((0,1);H^1_{\#}(Y^*)\big). \tag{1.2.14}$$

Therefore, adding up terms (1.2.13) and (1.2.14) we obtain the homogenized system (1.2.8).

We prove now the existence and uniqueness of solution for problem (1.2.8).

Note that, endowing the Hilbert space $H^1(0,1)\times L^2\Big((0,1);H^1_{\#}(Y^*)/\mathbb{R}\Big)$ with the following norm

$$\rho(\phi,\varphi) = \Big(\Big\|\frac{\partial \phi}{\partial x}+\frac{\partial \varphi}{\partial y_1}\Big\|^2_{L^2\big((0,1)\times Y^*\big)} + \Big\|\frac{\partial \varphi}{\partial y_2}\Big\|^2_{L^2\big((0,1)\times Y^*\big)} + \|\phi\|^2_{L^2\big((0,1)\times Y^*\big)}\Big)^{1/2},$$

$\forall\phi \in H^1(0,1)$, $\varphi \in L^2\Big((0,1); H^1_\#(Y^*)/\mathbb{R}\Big)$, it is obvious that (1.2.8) satisfies the conditions of the Lax-Milgram Theorem. Therefore, let us check that ρ defines a norm in $H^1(0,1) \times L^2\Big((0,1); H^1_\#(Y^*)/\mathbb{R}\Big)$. Since the other two properties are straightforward, let us focus on proving: $\rho(\phi, \varphi) = 0$ then $(\phi, \varphi) = 0$ in $H^1(0,1) \times L^2\Big((0,1); H^1_\#(Y^*)/\mathbb{R}\Big)$. Thus, assume that $\rho(\phi, \varphi) = 0$, that is,

$$\Big\| \frac{\partial \phi}{\partial x} + \frac{\partial \varphi}{\partial y_1} \Big\|^2_{L^2\left((0,1)\times Y^*\right)} + \Big\| \frac{\partial \varphi}{\partial y_2} \Big\|^2_{L^2\left((0,1)\times Y^*\right)} + \|\phi\|^2_{L^2\left((0,1)\times Y^*\right)} = 0.$$

Consequently, we obtain that

$$\|\phi\|^2_{L^2\left((0,1)\times Y^*\right)} = 0 \ \text{ and } \ \Big\| \frac{\partial \phi}{\partial x} + \frac{\partial \varphi}{\partial y_1} \Big\|^2_{L^2\left((0,1)\times Y^*\right)} + \Big\| \frac{\partial \varphi}{\partial y_2} \Big\|^2_{L^2\left((0,1)\times Y^*\right)} = 0,$$

which implies $\phi = 0$ and then, we get

$$\Big\| \frac{\partial \varphi}{\partial y_1} \Big\|^2_{L^2\left((0,1)\times Y^*\right)} + \Big\| \frac{\partial \varphi}{\partial y_2} \Big\|^2_{L^2\left((0,1)\times Y^*\right)} = 0.$$

As a consequence,

$$\frac{\partial \varphi}{\partial y_1}(x, y_1, y_2) = \frac{\partial \varphi}{\partial y_2}(x, y_1, y_2) = 0, \quad \text{a.e.}(x, y_1, y_2) \in (0,1) \times Y^*.$$

Then, since by hypothesis Y^* is connected we can conclude that φ depends only on x, see [57], which implies that $\varphi = 0$ in $L^2\Big((0,1); H^1_\#(Y^*)/\mathbb{R}\Big)$.

Therefore we have proved that ρ is a norm in $H^1(0,1) \times L^2\Big((0,1); H^1_\#(Y^*)/\mathbb{R}\Big)$. Moreover, we can easily prove that the norm ρ is equivalent to the usual norm of the product space. Hence, the product space $H^1(0,1) \times L^2\Big((0,1); H^1_\#(Y^*)/\mathbb{R}\Big)$ endowed with ρ is a Hilbert space. Consequently, there exists a unique solution of the homogenized system (1.2.8) in $H^1(0,1) \times L^2\Big((0,1); H^1_\#(Y^*)/\mathbb{R}\Big)$.

To conclude the proof we obtain the homogenized limit equation in terms of u. Taking into account (1.2.14) we have that u and u_1 satisfy

$$\begin{aligned} &\int_{(0,1)\times Y^*} \Big\{ \frac{\partial u_1}{\partial y_1}\frac{\partial \varphi}{\partial y_1} + \frac{\partial u_1}{\partial y_2}\frac{\partial \varphi}{\partial y_2} \Big\}\, dx dy_1 dy_2 \\ &\quad = -\int_{(0,1)\times Y^*} \frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial y_1} dx dy_1 dy_2,\ \forall \varphi \in L^2\big((0,1); H^1_\#(Y^*)\big). \end{aligned} \tag{1.2.15}$$

Moreover, using that u does not depend on y_1 and y_2 and since X is the unique L-periodic solution of the problem (1.2.10) we have that $-X(y_1, y_2)\frac{\partial u}{\partial x}(x)$ satisfies

$$-\frac{\partial u}{\partial x}\int_{Y^*} \nabla X \nabla \varphi\, dy_1 dy_2 = -\frac{\partial u}{\partial x}\int_{Y^*} \frac{\partial \varphi}{\partial y_1}\, dy_1 dy_2, \quad \forall \varphi \in H^1_\#(Y^*),$$

which implies that

$$\int_{(0,1)\times Y^*} \Big\{ -\frac{\partial u}{\partial x}\frac{\partial X}{\partial y_1}\frac{\partial \varphi}{\partial y_1} - \frac{\partial u}{\partial x}\frac{\partial X}{\partial y_2}\frac{\partial \varphi}{\partial y_2} \Big\}\, dx dy_1 dy_2$$

$$= \int_{(0,1)\times Y^*} -\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial y_1}\,dxdy_1dy_2,\ \forall \varphi \in L^2\big((0,1); H^1_{\#}(Y^*)\big). \tag{1.2.16}$$

Then, from (1.2.15) and (1.2.16) we write u_1 as follows

$$u_1(x, y_1, y_2) = -X(y_1, y_2)\frac{\partial u}{\partial x}(x),$$

where X is the unique L-periodic solution of (1.2.10).

Using this expression of u_1 in (1.2.13) we find that u has to satisfied

$$\int_{(0,1)\times Y^*} \left\{\Big(\frac{\partial u}{\partial x} - \frac{\partial X}{\partial y_1}\frac{\partial u}{\partial x}\Big)\frac{\partial \phi}{\partial x} + u\phi\right\} dxdy_1dy_2$$
$$= \int_{(0,1)\times Y^*} \hat{f}\phi\,dxdy_1dy_2 \quad \forall \phi \in H^1(0,1).$$

Equivalently, taking

$$q_0 = \frac{1}{|Y^*|}\int_{Y^*}\left\{1 - \frac{\partial X}{\partial y_1}(y_1, y_2)\right\}dy_1dy_2 \text{ and } f_0 = \frac{1}{|Y^*|}\int_{Y^*}\hat{f}dy_1dy_2$$

we get

$$\int_0^1 \left\{q_0\frac{\partial u}{\partial x}\frac{\partial \phi}{\partial x} + u\phi\right\} dx = \int_0^1 f_0\phi dx \quad \forall \phi \in H^1(0,1),$$

which is the weak formulation of (2.0.3).

It remains to prove existence and uniqueness of solution of the limit homogenized problem. However, it is an immediate consequence of Lax-Milgram Theorem once we see that the problem is well posed in the sense that $q_0 > 0$. For this we argue in a similar way as in [8]. Let $a(\cdot,\cdot)$ be the bilinear form $a(\cdot,\cdot)$ associated with the variational formulation of (1.2.10)

$$a(\Psi, \Phi) = \int_{Y^*} \nabla\Psi\cdot\nabla\Phi\,dy_1dy_2,$$

for $\Psi, \Phi \in H^1_{\#}(Y^*)$. Then, X satisfies

$$a(X, \Phi) = \int_{Y^*}\frac{\partial \Phi}{\partial y_1}\,dy_1dy_2,\ \text{ for any } \Phi \in H^1_{\#}(Y^*).$$

Consequently,

$$a(y_1 - X, \Phi) = \int_{Y^*}\frac{\partial \Phi}{\partial y_1}\,dy_1dy_2 - \int_{Y^*}\frac{\partial \Phi}{\partial y_1}\,dy_1dy_2 = 0,\ \text{ for any } \Phi \in H^1_{\#}(Y^*). \tag{1.2.17}$$

In particular, $a(y_1 - X, X) = 0$ Turning back to q_0, we have

$$q_0 = \frac{1}{|Y^*|}\int_{Y^*}\left\{1-\frac{\partial X}{\partial y_1}\right\}dy_1dy_2 = \frac{1}{|Y^*|}\int_{Y^*}\frac{\partial}{\partial y_1}\big(y_1-X\big)\frac{\partial y_1}{\partial y_1}dy_1dy_2 = \frac{1}{|Y^*|}a(y_1-X, y_1). \tag{1.2.18}$$

Hence, using (1.2.17) we get

$$|Y^*|q_0 = a(y_1 - X, y_1) - a(y_1 - X, -X) = a(y_1 - X, y_1 - X) = \|\nabla(y_1 - X)\|^2_{[L^2(Y^*)]^2}.$$

Therefore, since $|Y^*| > 0$ we can conclude that $q_0 > 0$. Indeed, if this is not true, then we would have

$$\frac{\partial(y_1 - X)}{\partial y_1} = 0, \qquad \frac{\partial(y_1 - X)}{\partial y_2} = 0$$

which implies that there exists a constant C such that $y_1 - X = C$. This is impossible because X is L-periodic in the first variable.

Since the Lax-Milgram Theorem guarantees the uniqueness of the weak solution of (1.2.9) we know that every weakly convergent subsequence of the sequence $\{u^\epsilon\}$ converges to the same limit. Thus the whole sequence $\{u^\epsilon\}$ converge weakly to the limit u. □

Remark 1.2.5. *Notice that since q_0 is constant it follows from standard elliptic regularity theory that $u \in H^2(0,1)$.*

Remark 1.2.6. *Observe that assuming extra regularity conditions on the function $g(\cdot)$, for instance $g \in C^1(\mathbb{R})$, an easy integration by parts shows that* (1.2.10) *is the variational formulation associated to the usual auxiliary problem defined in the basic cell*

$$\begin{cases} -\Delta X = 0 \text{ in } Y^*, \\ \dfrac{\partial X}{\partial N} = 0 \text{ on } B_2, \\ \dfrac{\partial X}{\partial N} = N_1 \text{ on } B_1, \\ \displaystyle\int_{Y^*} X \, dy_1 dy_2 = 0, \end{cases} \tag{1.2.19}$$

where $N = (N_1, N_2)$ is the unit outward normal to ∂Y^* and, B_1 and B_2 are the upper and the lower boundary of Y^* respectively.

Moreover, standard elliptic regularity theory shows that $X \in H^2(Y^*) \cap C^0(Y^*)$.

Remark 1.2.7. *Observe that in case the non homogeneous term does not depend on y, that is, $f^\epsilon(x,y) = f(x)$ with $f \in L^2(0,1)$, the limit Neumann problem is given by*

$$\begin{cases} -q_0 u_{xx} + u = f(x), \quad x \in (0,1) \\ \qquad u'(0) = u'(1) = 0 \end{cases}$$

where $q_0 = \dfrac{1}{|Y^*|} \displaystyle\int_{Y^*} \Big\{1 - \frac{\partial X}{\partial y_1}(y_1, y_2)\Big\} dy_1 dy_2.$

Remark 1.2.8. *Notice that the coefficient q_0 reflects how the geometry of the thin domain, in particular the rough boundary, affects the limit equation. In fact, we may prove that $q_0 < 1$. For this, we use* (1.2.17), (1.2.18) *and the basic properties of the symmetric bilinear form*

$$0 < |Y^*|q_0 = a(y_1 - X, y_1) = a(y_1 - X, y_1) + a(y_1 - X, X)$$

$$= a(y_1, y_1) + a(y_1, X) - a(X, y_1) - a(X, X)$$
$$= a(y_1, y_1) - a(X, X) = |Y^*| - \|\nabla(X)\|^2_{[L^2(Y^*)]^2} < |Y^*|.$$

Thus, we get $0 < q_0 < 1$.

We complete this section analyzing the strong convergence of the solutions without any additional regularity condition on the boundary of the thin domain. Actually, a convergence result stronger than (1.2.6) and (1.2.7) is obtained for the sequence of the gradient of the solutions u^ϵ. Moreover we get a general corrector result.

Proposition 1.2.9. *Assume that hypothesis of Theorem 1.2.4 are satisfied. Then, one has*

i) $\mathcal{T}_\epsilon(u^\epsilon) \overset{\epsilon \to 0}{\longrightarrow} u \quad s - L^p\big((0,1) \times Y^*\big)$ *and* $|||u^\epsilon - u|||_{L^2(R^\epsilon)} \overset{\epsilon \to 0}{\longrightarrow} 0.$

ii) The following strong convergences

$$\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big) \overset{\epsilon \to 0}{\longrightarrow} \frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial y_1} \quad s\text{–}L^2\big((0,1)\times Y^*\big),\ \mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial y}\Big) \overset{\epsilon \to 0}{\longrightarrow} \frac{\partial u_1}{\partial y_2} \quad s\text{–}L^2\big((0,1)\times Y^*\big).$$

Moreover,

$$\frac{1}{\epsilon}\int_{R_1^\epsilon} |\nabla u^\epsilon|^2 dxdy \overset{\epsilon \to 0}{\longrightarrow} 0,$$

where R_1^ϵ *is the subset of* R^ϵ *containing the corresponding part of the unique cell which is not totally included in* R^ϵ*, see* (1.1.3).

iii) Let X *be the unique solution of* (1.2.10)*. Then, the following convergence holds*

$$\lim_{\epsilon \to 0} |||\nabla u^\epsilon - \nabla u + \mathcal{U}_\epsilon\Big(\frac{\partial u}{\partial x}\Big)(X_1^\epsilon, X_2^\epsilon)|||_{[L^2(R^\epsilon)]^2} = 0,$$

where

$$X_1^\epsilon(x,y) \equiv \frac{\partial X}{\partial y_1}\big(\frac{x}{\epsilon}, \frac{y}{\epsilon}\big) \text{ and } X_2^\epsilon(x,y) \equiv \frac{\partial X}{\partial y_2}\big(\frac{x}{\epsilon}, \frac{y}{\epsilon}\big),\ \forall (x,y) \in R^\epsilon.$$

Proof. i) It is enough to apply Proposition 1.1.14 to the sequence of solutions $\{u^\epsilon\}$.

ii) The proof of these convergences is based on the convergence of the energy. Taking $\varphi = u^\epsilon$ in the variational formulation (1.0.3), we obtain

$$\int_{R^\epsilon} \Big\{\Big(\frac{\partial u^\epsilon}{\partial x}\Big)^2 + \Big(\frac{\partial u^\epsilon}{\partial y}\Big)^2\Big\}\, dxdy = \int_{R^\epsilon} \Big\{f^\epsilon u^\epsilon - (u^\epsilon)^2\Big\}\, dxdy.$$

Applying the unfolding operator and taking into account i) of this proposition we can pass to the limit to obtain

$$\begin{aligned} &\int_{(0,1)\times Y^*} \Big\{\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big)^2 + \mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial y}\Big)^2\Big\} dxdy_1dy_2 + \frac{1}{\epsilon}\int_{R_1^\epsilon} |\nabla u^\epsilon|^2 dxdy \\ &\quad \overset{\epsilon \to 0}{\longrightarrow} \int_{(0,1)\times Y^*} \Big\{\hat{f}u - u^2\Big\}\, dxdy_1dy_2. \end{aligned} \tag{1.2.20}$$

Now, considering $\phi = u$ and $\varphi = u_1$ as test functions in (1.2.8) we get

$$\int_{(0,1)\times Y^*} \left\{ \left(\frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial y_1} \right)^2 + \frac{\partial u_1}{\partial y_2}^2 \right\} dx dy_1 dy_2 = \int_{(0,1)\times Y^*} \left\{ \hat{f} u - u^2 \right\} dx dy_1 dy_2. \tag{1.2.21}$$

Finally, summing up (1.2.20) and (1.2.21) we have

$$\begin{aligned} &\int_{(0,1)\times Y^*} \left\{ \mathcal{T}_\epsilon \Big(\frac{\partial u^\epsilon}{\partial x} \Big)^2 + \mathcal{T}_\epsilon \Big(\frac{\partial u^\epsilon}{\partial y} \Big)^2 \right\} dx dy_1 dy_2 + \frac{1}{\epsilon} \int_{R_1^\epsilon} |\nabla u^\epsilon|^2 dx dy \\ &\qquad \stackrel{\epsilon \to 0}{\longrightarrow} \int_{(0,1)\times Y^*} \left\{ \left(\frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial y_1} \right)^2 + \frac{\partial u_1}{\partial y_2}^2 \right\} dx dy_1 dy_2. \end{aligned}$$

Therefore, using the limit above, weak convergences (1.2.6) and (1.2.7) and by standard weak lower-semicontinuity we obtain the following inequalities

$$\begin{aligned} &\int_{(0,1)\times Y^*} \left\{ \left(\frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial y_1} \right)^2 + \frac{\partial u_1}{\partial y_2}^2 \right\} dx dy_1 dy_2 \\ &\le \liminf_{\epsilon \to 0} \int_{(0,1)\times Y^*} \left\{ \mathcal{T}_\epsilon \Big(\frac{\partial u^\epsilon}{\partial x} \Big)^2 + \mathcal{T}_\epsilon \Big(\frac{\partial u^\epsilon}{\partial y} \Big)^2 \right\} dx dy_1 dy_2 \\ &\le \limsup_{\epsilon \to 0} \int_{(0,1)\times Y^*} \left\{ \mathcal{T}_\epsilon \Big(\frac{\partial u^\epsilon}{\partial x} \Big)^2 + \mathcal{T}_\epsilon \Big(\frac{\partial u^\epsilon}{\partial y} \Big)^2 \right\} dx dy_1 dy_2 \\ &\le \lim_{\epsilon \to 0} \left\{ \int_{(0,1)\times Y^*} \left\{ \mathcal{T}_\epsilon \Big(\frac{\partial u^\epsilon}{\partial x} \Big)^2 + \mathcal{T}_\epsilon \Big(\frac{\partial u^\epsilon}{\partial y} \Big)^2 \right\} dx dy_1 dy_2 \right. \\ &\quad \left. + \frac{1}{\epsilon} \int_{R_1^\epsilon} |\nabla u^\epsilon|^2 dx dy \right\} \\ &= \int_{(0,1)\times Y^*} \left\{ \left(\frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial y_1} \right)^2 + \frac{\partial u_1}{\partial y_2}^2 \right\} dx dy_1 dy_2. \end{aligned}$$

Consequently, we deduce that

$$\begin{aligned} &\int_{(0,1)\times Y^*} \left\{ \mathcal{T}_\epsilon \Big(\frac{\partial u^\epsilon}{\partial x} \Big)^2 + \mathcal{T}_\epsilon \Big(\frac{\partial u^\epsilon}{\partial y} \Big)^2 \right\} dx dy_1 dy_2 \\ &\qquad \stackrel{\epsilon \to 0}{\longrightarrow} \int_{(0,1)\times Y^*} \left\{ \left(\frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial y_1} \right)^2 + \frac{\partial u_1}{\partial y_2}^2 \right\} dx dy_1 dy_2. \qquad (1.2.22) \\ &\frac{1}{\epsilon} \int_{R_1^\epsilon} |\nabla u^\epsilon|^2 dx dy \stackrel{\epsilon \to 0}{\longrightarrow} 0. \end{aligned}$$

Hence, due to the weak convergences (1.2.6), (1.2.7) and the convergence (1.2.22) we obtain by the Radon-Riesz property the strong convergences of ii).

iii) From property v) of Proposition 1.1.16 and convergences obtained in ii) we immediately get

$$\lim_{\epsilon\to 0} |||\nabla u^\epsilon - \mathcal{U}_\epsilon(\nabla u) - \mathcal{U}_\epsilon(\nabla_{y_1y_2}u_1)|||_{[L^2(R^\epsilon)]^2} = 0.$$

Moreover, from iv) of Proposition 1.1.16 we have

$$|||\mathcal{U}_\epsilon(\nabla u) - \nabla u|||_{L^p(R^\epsilon)} \stackrel{\epsilon\to 0}{\longrightarrow} 0.$$

Consequently, we obtain

$$\lim_{\epsilon\to 0} |||\nabla u^\epsilon - \nabla u - \mathcal{U}_\epsilon(\nabla_{y_1y_2}u_1)|||_{[L^2(R^\epsilon)]^2} = 0.$$

Finally, since $u_1(x, y_1, y_2) = -X(y_1, y_2)\dfrac{\partial u}{\partial x}(x)$ the image by the averaging operator is given by

$$\mathcal{U}_\epsilon(\nabla_{y_1y_2}u_1) = -\mathcal{U}_\epsilon\Big(\frac{\partial u}{\partial x}\Big)\nabla_{y_1y_2}X(x/\epsilon, y/\epsilon).$$

Therefore, taking into account that by definition $\nabla_{y_1y_2}X(x/\epsilon, y/\epsilon) = (X_1^\epsilon, X_2^\epsilon)$ we get the desired convergence.

□

Corollary 1.2.10. *If $g(\cdot) \in C^1(\mathbb{R})$ then the following corrector result holds*

$$\lim_{\epsilon\to 0} \Big|\Big|\Big| u^\epsilon - u + \epsilon\frac{\partial u}{\partial x}X^\epsilon \Big|\Big|\Big|_{H^1(R^\epsilon)} = 0,$$

where $X^\epsilon(x, y) \equiv X(x/\epsilon, y/\epsilon)$, $(x, y) \in R^\epsilon$.

Proof. As we have mentioned in Remark 1.2.6 assuming that $g \in C^1(\mathbb{R})$ the function X is the unique L-periodic solution of problem (1.2.19). Thus, we have that $X \in H^2(Y^*) \cap C^0(Y^*)$.

First of all, notice that the corrector is well defined, that is, $\epsilon\frac{\partial u}{\partial x}X^\epsilon \in H^1(R^\epsilon)$ since $u \in H^2(0, 1)$ and $X \in H^2(Y^*)$. As a matter of fact, from property iv) of Proposition 1.1.4 we have that $X(x/\epsilon, y/\epsilon)$, $\frac{\partial X}{\partial y_1}(x/\epsilon, y/\epsilon)$ and $\frac{\partial X}{\partial y_2}(x/\epsilon, y/\epsilon)$ belong to $L^2(R^\epsilon)$. Moreover, their norms are uniformly bounded. The computation is similar for the three functions, we show the estimate for the first one

$$\begin{aligned}
|||X^\epsilon|||^2_{L^2(R^\epsilon)} &= \frac{1}{\epsilon}\int_{R^\epsilon} \Big|X\Big(\frac{x}{\epsilon}, \frac{y}{\epsilon}\Big)\Big|^2 dxdy \\
&\le \sum_{k=0}^{N_\epsilon+1} \int_{\epsilon kL}^{\epsilon L(k+1)} \int_0^{\epsilon g(x/\epsilon)} \Big|X\Big(\frac{x}{\epsilon}, \frac{y}{\epsilon}\Big)\Big|^2 dydx \\
&= \sum_{k=0}^{N_\epsilon+1} \epsilon \int_{Y^*} |X(y_1, y_2)|^2 dy_1dy_2
\end{aligned}$$

$$= C \int_{Y^*} |X|^2 dy_1 dy_2 = C||X||^2_{L^2(Y^*)}.$$

Using the same notation as in Proposition 1.2.9 we consider

$$X_1^\epsilon(x,y) \equiv \frac{\partial X}{\partial y_1}(\frac{x}{\epsilon},\frac{y}{\epsilon}) \text{ and } X_2^\epsilon(x,y) \equiv \frac{\partial X}{\partial y_2}(\frac{x}{\epsilon},\frac{y}{\epsilon}),\ \forall (x,y) \in R^\epsilon.$$

By the definition of the norm $|||\cdot|||_{H^1(R^\epsilon)}$ we obtain

$$\begin{aligned} \Big|\Big|\Big| u^\epsilon - u + \epsilon\frac{\partial u}{\partial x}X^\epsilon \Big|\Big|\Big|^2_{H^1(R^\epsilon)} &= \Big|\Big|\Big| u^\epsilon - u + \epsilon\frac{\partial u}{\partial x}X^\epsilon \Big|\Big|\Big|^2_{L^2(R^\epsilon)} \\ + \Big|\Big|\Big| \frac{\partial u^\epsilon}{\partial x} - \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}X_1^\epsilon + \epsilon\frac{\partial^2 u}{\partial x^2}X^\epsilon \Big|\Big|\Big|^2_{L^2(R^\epsilon)} \\ + \Big|\Big|\Big| \frac{\partial u^\epsilon}{\partial y} + \frac{\partial u}{\partial x}X_2^\epsilon \Big|\Big|\Big|^2_{L^2(R^\epsilon)}, \end{aligned} \tag{1.2.23}$$

Now, we calculate the limit for each term of the right-hand side of (1.2.23). For the first term using the triangular inequality we have

$$\Big|\Big|\Big| u^\epsilon - u + \epsilon\frac{\partial u}{\partial x}X^\epsilon \Big|\Big|\Big|_{L^2(R^\epsilon)} \leq |||u^\epsilon - u|||_{L^2(R^\epsilon)} + \Big|\Big|\Big| \epsilon\frac{\partial u}{\partial x}X^\epsilon \Big|\Big|\Big|_{L^2(R^\epsilon)}. \tag{1.2.24}$$

Observe that, since $X \in H^1(Y^*)$ and $u \in H^2(0,1)$ we get

$$\Big|\Big|\Big| \epsilon\frac{\partial u}{\partial x}X^\epsilon \Big|\Big|\Big|_{L^2(R^\epsilon)} \leq \epsilon \Big|\Big| \frac{\partial u}{\partial x} \Big|\Big|_{L^\infty(0,1)} |||X^\epsilon|||_{L^2(R^\epsilon)} \leq C\epsilon. \tag{1.2.25}$$

Then, from convergence i) of Proposition 1.2.9, estimate (1.2.25) and inequality (1.2.24) we obtain that the first term of (1.2.23) tends to zero, that is,

$$\Big|\Big|\Big| u^\epsilon - u + \epsilon\frac{\partial u}{\partial x}X^\epsilon \Big|\Big|\Big|^2_{L^2(R^\epsilon)} \stackrel{\epsilon\to 0}{\longrightarrow} 0. \tag{1.2.26}$$

For the second term of the right-hand side of (1.2.23), adding and subtracting appropriate functions and using the triangular inequality we get

$$\begin{aligned} &\Big|\Big|\Big| \frac{\partial u^\epsilon}{\partial x} - \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}X_1^\epsilon + \epsilon\frac{\partial^2 u}{\partial x^2}X^\epsilon \Big|\Big|\Big|_{L^2(R^\epsilon)} \\ &\leq \Big|\Big|\Big| \frac{\partial u^\epsilon}{\partial x} - \frac{\partial u}{\partial x} + \mathcal{U}_\epsilon\Big(\frac{\partial u}{\partial x}\Big)X_1^\epsilon \Big|\Big|\Big|_{L^2(R^\epsilon)} \\ &+ \Big|\Big|\Big| \frac{\partial u}{\partial x}X_1^\epsilon - \mathcal{U}_\epsilon\Big(\frac{\partial u}{\partial x}\Big)X_1^\epsilon \Big|\Big|\Big|_{L^2(R^\epsilon)} + \Big|\Big|\Big| \epsilon\frac{\partial^2 u}{\partial x^2}X^\epsilon \Big|\Big|\Big|_{L^2(R^\epsilon)}. \end{aligned} \tag{1.2.27}$$

From convergence iii) of Proposition 1.2.9 follows that

$$\Big|\Big|\Big| \frac{\partial u^\epsilon}{\partial x} - \frac{\partial u}{\partial x} + \mathcal{U}_\epsilon\Big(\frac{\partial u}{\partial x}\Big)X_1^\epsilon \Big|\Big|\Big|_{L^2(R^\epsilon)} \stackrel{\epsilon\to 0}{\longrightarrow} 0. \tag{1.2.28}$$

From property iv) of Proposition 1.1.4 we have

$$\mathcal{T}_\epsilon(X_1^\epsilon)(x, y_1, y_2) = \frac{\partial X}{\partial y_1}(y_1, y_2), \quad \forall (x, y_1, y_2) \in I^\epsilon \times Y^*.$$

Moreover in view of the proof of Proposition 1.1.9 we get

$$\frac{1}{\epsilon}\int_{R_1^\epsilon} |X_1^\epsilon|^2 \, dxdy = \epsilon \int_{Y^*} |\frac{\partial X}{\partial y_1}(y_1, y_2)|^2 dy_1 dy_2. \tag{1.2.29}$$

Therefore, we have

$$\mathcal{T}_\epsilon(X_1^\epsilon) \overset{\epsilon\to 0}{\longrightarrow} \frac{\partial X}{\partial y_1} \qquad \text{s} - L^2\big((0,1) \times Y^*\big).$$

As a consequence, taking into account Proposition 1.1.11 we obtain

$$\mathcal{T}_\epsilon\Big(\frac{\partial u}{\partial x} X_1^\epsilon\Big) \overset{\epsilon\to 0}{\longrightarrow} \frac{\partial u}{\partial x}\frac{\partial X}{\partial y_1} \qquad \text{s} - L^2\big((0,1) \times Y^*\big).$$

Then, since $\mathcal{U}_\epsilon\Big(\frac{\partial u}{\partial x}\frac{\partial X}{\partial y_1}\Big) = \mathcal{U}_\epsilon\Big(\frac{\partial u}{\partial x}\Big) X_1^\epsilon$, $\frac{\partial u}{\partial x} \in L^\infty(0,1)$ and X_1^ϵ satisfies (1.2.29) it follows from property v) of Proposition 1.1.16 that

$$\Big|\Big|\Big|\frac{\partial u}{\partial x} X_1^\epsilon - \mathcal{U}_\epsilon\Big(\frac{\partial u}{\partial x}\Big) X_1^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0. \tag{1.2.30}$$

In addition, we obtain

$$\Big|\Big|\Big|\epsilon \frac{\partial^2 u}{\partial x^2} X^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)} \leq \epsilon ||X||_{L^\infty(Y^*)} \Big|\Big|\Big|\frac{\partial^2 u}{\partial x^2}\Big|\Big|\Big|_{L^2(R^\epsilon)} \leq C\epsilon,$$

which implies that

$$\Big|\Big|\Big|\epsilon \frac{\partial^2 u}{\partial x^2} X^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0. \tag{1.2.31}$$

Therefore, in view of (1.2.27), (1.2.28), (1.2.30) and (1.2.31) we get

$$\Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial x} - \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} X_1^\epsilon + \epsilon \frac{\partial^2 u}{\partial x^2} X^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0. \tag{1.2.32}$$

By repeating a similar reasoning for the third term of (1.2.23) we obtain

$$\Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial y} + \frac{\partial u}{\partial x} X_2^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0. \tag{1.2.33}$$

Therefore, thanks to convergences (1.2.26), (1.2.32), (1.2.33) and the inequality (1.2.23) we get the desired convergence

$$\lim_{\epsilon\to 0} \Big|\Big|\Big|u^\epsilon - u + \epsilon \frac{\partial u}{\partial x} X^\epsilon\Big|\Big|\Big|_{H^1(R^\epsilon)} = 0.$$

□

Remark 1.2.11. *Observe that Corollary 1.2.10 shows that assuming extra regularity conditions, $g \in C^1(\mathbb{R})$, we get the standard corrector result proved in [96].*

1.3. Weak oscillations, $\alpha < 1$

In this section we show how the unfolding operator introduced in Definition 1.1.3 can be used to study the behavior of the solutions of (1.0.2) for the case of weak roughness. Then, we are dealing now with an oscillatory thin domain R^ϵ defined as

$$R^\epsilon = \Big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ 0 < y < \epsilon\, g(x/\epsilon^\alpha)\Big\},$$

where the oscillatory boundary is described by a periodic function with period of order ϵ^α and amplitude of order ϵ. We require that $g(\cdot)$ satisfies hypothesis **Hg)** stated at the beginning of Section 1.1. Moreover we assume that $0 < g_0$ in order to guarantee that the Poincaré−Wirtinger inequality holds in the representative cell Y^*. Then, we say that the thin domains present weak oscillations because the order of the thickness of the domain is larger than the order of the frequency of the oscillations. The ratio $\dfrac{\epsilon}{\epsilon^\alpha}$ converges to zero since $0 < \alpha < 1$.

To obtain the homogenized limit problem we will follow a similar approach as in the previous section. Then, we begin this section with the corresponding compactness result.

Theorem 1.3.1. *Let $\varphi^\epsilon \in W^{1,p}(R^\epsilon)$ for some $1 < p < \infty$, with $|||\varphi^\epsilon|||_{W^{1,p}(R^\epsilon)}$ uniformly bounded. Then, there exist functions $\varphi \in W^{1,p}(0,1)$, $\varphi_1 \in L^p\big((0,1); W^{1,p}_\#(Y^*)\big)$ with $\dfrac{\partial \varphi_1}{\partial y_2} = 0$ such that, up to subsequences*

i) $\mathcal{T}_\epsilon(\varphi^\epsilon) \stackrel{\epsilon\to 0}{\rightharpoonup} \varphi \qquad w - L^p\big((0,1); W^{1,p}(Y^*)\big)$,

ii) $\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial x}\Big) \stackrel{\epsilon\to 0}{\rightharpoonup} \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi_1}{\partial y_1} \qquad w - L^p\big((0,1)\times Y^*\big)$.

Proof. i) This convergence was obtained in Proposition 1.1.14 for any α greater than 0.

ii) This assertion can be proved following the same arguments as the corresponding proof of Theorem 1.2.3. Therefore, we stress here just the main differences respect to the case $\alpha = 1$. We consider the operator

$$Z_\epsilon := \frac{1}{\epsilon^\alpha}\Big(\mathcal{T}_\epsilon(\varphi^\epsilon) - \frac{1}{|Y^*|}\int_{Y^*} T_\epsilon(\varphi^\epsilon)\, dy_2 dy_1\Big), \tag{1.3.1}$$

which has mean value zero in Y^* and from Proposition 1.1.4 vii) satisfies

$$\begin{aligned} \frac{\partial Z_\epsilon}{\partial y_1} &= \mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial x}\Big), \\ \frac{\partial Z_\epsilon}{\partial y_2} &= \epsilon^{1-\alpha}\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial y}\Big). \end{aligned} \tag{1.3.2}$$

In the same way as the resonant case it is not difficult to prove that there exists a function $\varphi_1 \in L^p\big((0,1); W^{1,p}(Y^*)\big)$ such that, up to subsequences,

$$Z_\epsilon - y_1^c \frac{\partial \varphi}{\partial x} \stackrel{\epsilon\to 0}{\rightharpoonup} \varphi_1 \quad \mathrm{w} - L^p\big((0,1); W^{1,p}(Y^*)\big), \tag{1.3.3}$$

where $y_1^c = y_1 - \frac{1}{|Y^*|}\int_{Y^*} y_1\, dy_2 dy_1$.

Hence, taking into account (1.3.2) we get

$$\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial x}\Big) \overset{\epsilon\to 0}{\rightharpoonup} \frac{\partial \varphi}{\partial x}(x) + \frac{\partial \varphi_1}{\partial y_1}(x, y_1, y_2) \quad \mathrm{w} - L^p\big((0,1)\times Y^*\big).$$

However, unlike the previous case, the function φ_1 does not depend on y_2. To see this recall that the partial derivative respect y_2 of Z_ϵ is of the form

$$\frac{\partial Z_\epsilon}{\partial y_2} = \epsilon^{1-\alpha}\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial y}\Big).$$

Since $1-\alpha > 0$ and $\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial y}\Big)$ is bounded we obtain that

$$\frac{\partial Z_\epsilon}{\partial y_2} \overset{\epsilon\to 0}{\longrightarrow} 0 \quad \mathrm{s} - L^2\big((0,1)\times Y^*\big).$$

Then, from (1.3.3) and the uniqueness of the limit we obtain $\dfrac{\partial \varphi_1}{\partial y_2} = 0$. Thus, φ_1 is independent of y_2.

Finally, by the same computation as Theorem 1.2.3 we get the periodicity of φ_1.

□

Remark 1.3.2. *Notice that, besides the convergences obtained in Theorem 1.3.1 we have from Proposition 1.1.14 that there exists $\bar\varphi \in L^p((0,1)\times Y^*)$ with $\frac{\partial \bar\varphi}{\partial y_2} \in L^p((0,1)\times Y^*)$ such that, up to subsequences*

$$\mathcal{T}_\epsilon\big(\frac{\partial \varphi^\epsilon}{\partial y}\big) \overset{\epsilon\to 0}{\rightharpoonup} \frac{\partial \bar\varphi}{\partial y_2} \quad w\text{-}L^p((0,1)\times Y^*).$$

However, we would like to point out that we will not need this convergence in order to get the homogenized limit problem, see proof of Theorem 1.3.3.

Now we are in conditions to get the homogenized limit for problem (1.0.2) when $0 < \alpha < 1$.

Theorem 1.3.3. *Let u^ϵ be the solution of problem (1.0.3) with $f^\epsilon \in L^2(R^\epsilon)$ satisfying $|||f^\epsilon|||_{L^2(R^\epsilon)} \le C$ for some positive constant C independent of the parameter $\epsilon > 0$. Assume that there exists $\hat f \in L^2((0,1)\times Y^*)$ such that*

$$\mathcal{T}_\epsilon(f^\epsilon) \overset{\epsilon\to 0}{\rightharpoonup} \hat f \text{ weakly in } L^2\big((0,1)\times Y^*\big).$$

Then, there exist $u \in H^1(0,1)$ and $u_1 \in L^2\Big((0,1); H^1_\#\big(Y^*\big)\Big)$ with $\dfrac{\partial u_1}{\partial y_2} = 0$ such that

$$\mathcal{T}_\epsilon(u^\epsilon) \overset{\epsilon\to 0}{\rightharpoonup} u \quad \text{weakly in } L^2\big((0,1); H^1(Y^*)\big),$$

$$\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big) \overset{\epsilon\to 0}{\rightharpoonup} \frac{\partial u}{\partial x}(x) + \frac{\partial u_1}{\partial y_1}(x, y_1, y_2) \quad \textit{weakly in} \;\; L^2\big((0,1)\times Y^*\big), \tag{1.3.4}$$

where u_1 is the unique function, up to constants, such that

$$\frac{\partial u_1}{\partial y_1} = \Big(-1 + \frac{1}{g\mathcal{M}(\frac{1}{g})}\Big)\frac{\partial u}{\partial x},$$

and $u \in H^1(0,1)$ is the unique solution of the Neumann problem

$$\begin{cases} -\dfrac{1}{\mathcal{M}(g)\mathcal{M}\big(\frac{1}{g}\big)} u_{xx} + u = f_0(x), \quad x \in (0,1), \\ \qquad\qquad u'(0) = u'(1) = 0, \end{cases} \tag{1.3.5}$$

where $f_0(x) = \dfrac{1}{|Y^*|}\displaystyle\int_{Y^*} \hat f(x, y_1, y_2)dy_1dy_2.$

Remark 1.3.4. *Observe that the limit problem is well defined since the assumptions on the function g imply that $\dfrac{1}{g} \in L^1(0,1)$.*

Proof. We start by establishing a priori estimates of u^ϵ, solution of (1.0.3). Taking u^ϵ as a test function in (1.0.3) we get

$$\Big\|\frac{\partial u^\epsilon}{\partial x}\Big\|^2_{L^2(R^\epsilon)} + \Big\|\frac{\partial u^\epsilon}{\partial y}\Big\|^2_{L^2(R^\epsilon)} + \|u^\epsilon\|^2_{L^2(R^\epsilon)} \le \|f^\epsilon\|_{L^2(R^\epsilon)}\|u^\epsilon\|_{L^2(R^\epsilon)},$$

which implies that

$$\epsilon^{-1}\Big\|\frac{\partial u^\epsilon}{\partial x}\Big\|^2_{L^2(R^\epsilon)} + \epsilon^{-1}\Big\|\frac{\partial u^\epsilon}{\partial y}\Big\|^2_{L^2(R^\epsilon)} + \epsilon^{-1}\|u^\epsilon\|^2_{L^2(R^\epsilon)} \le \epsilon^{-1/2}\|f^\epsilon\|_{L^2(R^\epsilon)}\epsilon^{-1/2}\|u^\epsilon\|_{L^2(R^\epsilon)}.$$

Since $|||f^\epsilon|||_{L^2(R^\epsilon)} \le C$, with C independent of ϵ, we deduce that

$$|||u^\epsilon|||_{H^1(R^\epsilon)} \le C.$$

From Theorem 1.3.1 there are two functions $u \in H^1(0,1)$ and $u_1 \in L^2\big((0,1); H^1_\#(Y^*)\big)$ with $\frac{\partial u_1}{\partial y_2} = 0$ such that, up to subsequences,

$$\begin{aligned} &\mathcal{T}_\epsilon(u^\epsilon) \overset{\epsilon\to 0}{\rightharpoonup} u \quad \text{weakly in} \;\; L^2\big((0,1); H^1(Y^*)\big), \\ &\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big) \overset{\epsilon\to 0}{\rightharpoonup} \frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial y_1} \quad \text{weakly in} \;\; L^2((0,1)\times Y^*). \end{aligned} \tag{1.3.6}$$

We are now in position to provide the equations satisfied by u and u_1. Transforming (1.0.3) by the unfolding operator $\mathcal{T}_\epsilon$ we have

$$\int_{(0,1)\times Y^*} \Big\{\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big)\mathcal{T}_\epsilon\Big(\frac{\partial \phi}{\partial x}\Big) + \mathcal{T}_\epsilon(u^\epsilon)\mathcal{T}_\epsilon(\phi)\Big\} dxdy_1dy_2$$

$$
\begin{aligned}
&+\frac{L}{\epsilon}\int_{R_1^\epsilon}\left\{\frac{\partial u^\epsilon}{\partial x}\frac{\partial \phi}{\partial x}+u^\epsilon\phi\right\}dxdy\\
&=\int_{(0,1)\times Y^*}\mathcal{T}_\epsilon(f^\epsilon)\mathcal{T}_\epsilon(\phi)dxdy_1dy_2+\frac{L}{\epsilon}\int_{R_1^\epsilon}f^\epsilon\phi\,dxdy,\ \ \forall\phi\in H^1(0,1),
\end{aligned}
$$

which taking into account convergences (1.3.6) and Proposition 1.1.8 gives at the limit

$$
\begin{aligned}
&\int_{(0,1)\times Y^*}\left\{\Big(\frac{\partial u}{\partial x}(x)+\frac{\partial u_1}{\partial y_1}(x,y_1)\Big)\frac{\partial\phi}{\partial x}(x)+u(x)\phi(x)\right\}dxdy_1dy_2\\
&\qquad=\int_{(0,1)\times Y^*}\hat{f}(x,y_1,y_2)\phi(x)dxdy_1dy_2\quad\forall\phi\in H^1(0,1).
\end{aligned}
\tag{1.3.7}
$$

To identify u_1, we introduce the function $v^\epsilon\in H^1(R^\epsilon)$ defined by

$$
v^\epsilon(x,y)=\epsilon^\alpha\phi(x)\psi(x/\epsilon^\alpha)
$$

where $\phi\in\mathcal{D}(0,1)$ and $\psi\in H^1_\#(Y^*)$ such that $\frac{\partial\psi}{\partial y_2}=0$, that is, $\psi(y_1,y_2)=\psi(y_1)$ and $\psi(0)=\psi(L)$. It is easy to get the partial derivatives

$$
\frac{\partial v^\epsilon}{\partial x}=\epsilon^\alpha\frac{\partial\phi}{\partial x}(x)\psi\Big(\frac{x}{\epsilon^\alpha}\Big)+\phi(x)\frac{\partial\psi}{\partial y_1}\Big(\frac{x}{\epsilon^\alpha}\Big),\qquad \frac{\partial v^\epsilon}{\partial y}=0.
$$

Thus, using the basic properties of the unfolding operator we easily get

$$
\begin{aligned}
&\mathcal{T}_\epsilon(v^\epsilon)\xrightarrow{\epsilon\to 0}0\quad \text{s-}L^2((0,1)\times Y^*),\\
&\mathcal{T}_\epsilon\Big(\frac{\partial v^\epsilon}{\partial x}\Big)\xrightarrow{\epsilon\to 0}\phi\frac{\partial\psi}{\partial y_1}\quad \text{s-}L^2((0,1)\times Y^*),\\
&\mathcal{T}_\epsilon\Big(\frac{\partial v^\epsilon}{\partial y}\Big)=0.
\end{aligned}
\tag{1.3.8}
$$

We take now $v^\epsilon\in H^1(R^\epsilon)$ as a test function in (1.0.3)

$$
\int_{R^\epsilon}\left\{\frac{\partial u^\epsilon}{\partial x}\frac{\partial v^\epsilon}{\partial x}+u^\epsilon v^\epsilon\right\}dxdy=\int_{R^\epsilon}f^\epsilon v^\epsilon dxdy.
$$

Then, applying the unfolding operator leads to

$$
\begin{aligned}
&\int_{(0,1)\times Y^*}\left\{\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big)\mathcal{T}_\epsilon\Big(\frac{\partial v^\epsilon}{\partial x}\Big)+\mathcal{T}_\epsilon(u^\epsilon)\mathcal{T}_\epsilon(v^\epsilon)\right\}dxdy_1dy_2+\frac{L}{\epsilon}\int_{R_1^\epsilon}\left\{\frac{\partial u^\epsilon}{\partial x}\frac{\partial v^\epsilon}{\partial x}+u^\epsilon v^\epsilon\right\}dxdy\\
&\qquad=\int_{(0,1)\times Y^*}\mathcal{T}_\epsilon(f^\epsilon)\mathcal{T}_\epsilon(v^\epsilon)dxdy_1dy_2+\frac{L}{\epsilon}\int_{R_1^\epsilon}f^\epsilon v^\epsilon dxdy,
\end{aligned}
\tag{1.3.9}
$$

Convergences (1.3.6), (1.3.8) and Proposition 1.1.9 allows us to pass to the limit and we obtain

$$
\int_{(0,1)\times Y^*}\Big(\frac{\partial u}{\partial x}(x)+\frac{\partial u_1}{\partial y_1}(x,y_1)\Big)\phi(x)\frac{\partial\psi}{\partial y_1}(y_1)\,dxdy_1dy_2=0,
$$

for any $\phi \in \mathcal{D}(0,1)$ and $\psi \in H^1_\#(Y^*)$ such that $\frac{\partial \psi}{\partial y_2} = 0$. By density, this equality holds true for all $\psi \in L^2\big((0,1); H^1_\#(Y^*)\big)$ with $\frac{\partial \psi}{\partial y_2} = 0$

$$\int_{(0,1)\times Y^*} \Big(\frac{\partial u}{\partial x}(x) + \frac{\partial u_1}{\partial y_1}(x,y_1)\Big)\frac{\partial \psi}{\partial y_1}(x,y_1)\, dxdy_1dy_2 = 0. \tag{1.3.10}$$

Since all functions in (1.3.10) do not depend on y_2 we have

$$\int_{(0,1)\times(0,L)} \Big(\frac{\partial u}{\partial x}(x) + \frac{\partial u_1}{\partial y_1}(x,y_1)\Big)g(y_1)\frac{\partial \psi}{\partial y_1}(x,y_1)\, dxdy_1 = 0.$$

Hence, treating x as a parameter in the above equation we have that there exists a function T depending only on x such that

$$\Big(\frac{\partial u}{\partial x}(x) + \frac{\partial u_1}{\partial y_1}(x,y_1)\Big)g(y_1) = T(x) \qquad \text{a.e. in } (0,L).$$

Consequently,

$$\frac{\partial u_1}{\partial y_1} = -\frac{\partial u}{\partial x} + \frac{T}{g}.$$

Moreover, since u_1 is L-periodic we have

$$0 = \frac{1}{L}\int_{(0,L)} \frac{\partial u_1}{\partial y_1}\, dy_1 = -\frac{\partial u}{\partial x} + \frac{T}{L}\int_{(0,L)} \frac{1}{g}\, dy_1 = -\frac{\partial u}{\partial x} + T\mathcal{M}\Big(\frac{1}{g}\Big).$$

Then, we get

$$\frac{\partial u_1}{\partial y_1} = \Big(-1 + \frac{1}{g\mathcal{M}\big(\frac{1}{g}\big)}\Big)\frac{\partial u}{\partial x}.$$

Replacing u_1 by its value in the equation (1.3.7) we obtain the weak formulation of (1.3.5)

$$\int_0^1 \Big\{\frac{1}{\mathcal{M}(g)\mathcal{M}\big(\frac{1}{g}\big)}\, u_x(x)\,\phi_x(x) + u(x)\,\phi(x)\Big\}dx = \int_0^1 f_0(x)\,\phi(x)\,dx, \quad \forall \varphi \in H^1(0,1).$$

Thanks to Lax-Milgram Theorem we can ensure the existence and uniqueness of solution of (1.3.5) which ends the proof. □

Remark 1.3.5. *Observe that if the non homogeneous term is a fixed function depending only on x, $f^\epsilon(x,y) = f(x)$ with $f \in L^2(0,1)$, then the limit problem is given by*

$$\begin{cases} -\dfrac{1}{\mathcal{M}(g)\mathcal{M}\big(\frac{1}{g}\big)}u_{xx} + u = f(x), & x \in (0,1), \\ u'(0) = u'(1) = 0, \end{cases}$$

Now we are going to get new strong convergences which improve the convergences obtained in Theorem 1.3.3 without additional assumptions. We give a corrector result which shows the strong convergence for the gradient of the solutions u^ϵ to the derivative of u adding the first corrector function.

Proposition 1.3.6. *Assume that hypothesis of Theorem 1.3.3 are satisfied. Then, the following strong convergences hold*

i) $\mathcal{T}_\epsilon(u^\epsilon) \overset{\epsilon\to 0}{\longrightarrow} u \;\; s - L^2\big((0,1)\times Y^*\big)$ *and* $|||u^\epsilon - u|||_{L^2(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0.$

ii) $\mathcal{T}_\epsilon\Big(\dfrac{\partial u^\epsilon}{\partial x}\Big) \overset{\epsilon\to 0}{\longrightarrow} \dfrac{\partial u}{\partial x} + \dfrac{\partial u_1}{\partial y_1} \;\; s - L^2\big((0,1)\times Y^*\big),\; \mathcal{T}_\epsilon\Big(\dfrac{\partial u^\epsilon}{\partial y}\Big) \overset{\epsilon\to 0}{\longrightarrow} 0 \;\; s - L^2\big((0,1)\times Y^*\big).$

Moreover,

$$\frac{1}{\epsilon}\int_{R_1^\epsilon} |\nabla u^\epsilon|^2 dxdy \overset{\epsilon\to 0}{\longrightarrow} 0.$$

iii) $\lim\limits_{\epsilon\to 0} |||\nabla u^\epsilon - \nabla u - \mathcal{U}_\epsilon(\nabla_{y_1y_2} u_1)|||_{[L^2(R^\epsilon)]^2} = 0.$

iv) Let X be a function in $H^1_\#(Y^*)$ satisfying

$$\frac{\partial X}{\partial y_1} = 1 - \frac{1}{g\mathcal{M}(\frac{1}{g})} \qquad \textit{and} \qquad \frac{\partial X}{\partial y_2} = 0.$$

Then,

$$\lim_{\epsilon\to 0}\Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial x} - \frac{\partial u}{\partial x} + \mathcal{U}_\epsilon\Big(\frac{\partial u}{\partial x}\Big) X_1^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)} = 0,$$
$$\lim_{\epsilon\to 0}\Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial y}\Big|\Big|\Big|_{L^2(R^\epsilon)} = 0,$$

where $X_1^\epsilon(x) \equiv \dfrac{\partial X}{\partial y_1}\Big(\dfrac{x}{\epsilon^\alpha}\Big)$, $x \in (0,1)$.

Proof. i) It follows directly from Proposition 1.1.14.

ii) These convergences are based on the convergence of the energy. Taking u^ϵ as a test function in (1.0.3) we obtain

$$\int_{R^\epsilon} \Big\{\Big(\frac{\partial u^\epsilon}{\partial x}\Big)^2 + \Big(\frac{\partial u^\epsilon}{\partial y}\Big)^2\Big\} dxdy = \int_{R^\epsilon} \Big\{f^\epsilon u^\epsilon - (u^\epsilon)^2\Big\} dxdy.$$

Applying the unfolding operator and passing to the limit we have

$$\int_{(0,1)\times Y^*} \Big\{\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big)^2 + \mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial y}\Big)^2\Big\} dxdy_1dy_2 + \frac{1}{\epsilon}\int_{R_1^\epsilon} |\nabla u^\epsilon|^2 dxdy$$
$$\overset{\epsilon\to 0}{\longrightarrow} \int_{(0,1)\times Y^*} \Big\{\hat{f} u - u^2\Big\} dxdy_1dy_2. \quad (1.3.11)$$

On the other hand, taking u as a test function in (1.3.7) and u_1 as a test function in (1.3.10) we get

$$\int_{(0,1)\times Y^*} \Big\{\Big(\frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial y_1}\Big)\frac{\partial u}{\partial x} + u^2\Big\} dxdy_1dy_2 = \int_{(0,1)\times Y^*} \hat{f} u dxdy_1dy_2.$$

$$\int_{(0,1)\times Y^*} \Big(\frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial y_1}\Big)\frac{\partial u_1}{\partial y_1}\, dxdy_1dy_2 = 0.$$

Consequently we have

$$\int_{(0,1)\times Y^*} \Big\{\Big(\frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial y_1}\Big)^2 + u^2\Big\}\, dxdy_1dy_2 = \int_{(0,1)\times Y^*} \hat{f}u\, dxdy_1dy_2. \quad (1.3.12)$$

Therefore, combining (1.3.11) and (1.3.12) we obtain

$$\int_{(0,1)\times Y^*} \Big\{\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big)^2 + \mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial y}\Big)^2\Big\} dxdy_1dy_2 + \frac{1}{\epsilon}\int_{R_1^\epsilon} |\nabla u^\epsilon|^2\, dxdy$$
$$\xrightarrow{\epsilon\to 0} \int_{(0,1)\times Y^*} \Big(\frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial y_1}\Big)^2 dxdy_1dy_2$$

Therefore, from standard weak lower-semicontinuity we have

$$\begin{aligned}
&\int_{(0,1)\times Y^*} \Big(\frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial y_1}\Big)^2 dxdy_1dy_2 \\
&\leq \liminf_{\epsilon\to 0} \int_{(0,1)\times Y^*} \mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big)^2 dxdy_1dy_2 \\
&\leq \limsup_{\epsilon\to 0} \int_{(0,1)\times Y^*} \mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big)^2 dxdy_1dy_2 \\
&\leq \lim_{\epsilon\to 0}\Big\{ \int_{(0,1)\times Y^*} \Big\{\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big)^2 + \mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial y}\Big)^2\Big\} dxdy_1dy_2 \\
&\quad + \frac{1}{\epsilon}\int_{R_1^\epsilon} |\nabla u^\epsilon|^2\Big\} dxdy \\
&= \int_{(0,1)\times Y^*} \Big(\frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial y_1}\Big)^2 dxdy_1dy_2,
\end{aligned}$$

which gives the following convergences

$$\lim_{\epsilon\to 0} \int_{(0,1)\times Y^*} \mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big)^2 dxdy_1dy_2 = \int_{(0,1)\times Y^*} \Big(\frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial y_1}\Big)^2 dxdy_1dy_2, \quad (1.3.13)$$

$$\lim_{\epsilon\to 0} \int_{(0,1)\times Y^*} \mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial y}\Big)^2 dxdy_1dy_2 = 0,$$

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_{R_1^\epsilon} |\nabla u^\epsilon|^2 dxdy = 0. \quad (1.3.14)$$

Finally, in view of the the weak convergence (1.3.4) and the convergence of the norm (1.3.13) we have by the Radon-Riesz property the strong convergences of ii).

iii) It is a direct consequence from the convergences obtained in ii) and the properties iv) and v) of Proposition 1.1.16.

iv) Using the usual notation referred to corrector results we introduce an $L-$periodic function $X \in H^1_\#(Y^*)$ such that $u_1 = -X\dfrac{\partial u}{\partial x}$. Then, X satisfies

$$\frac{\partial X}{\partial y_1} = 1 - \frac{1}{g\mathcal{M}(\frac{1}{g})} \quad \text{and} \quad \frac{\partial X}{\partial y_2} = 0.$$

Moreover,

$$\mathcal{U}_\epsilon\Big(\frac{\partial u_1}{\partial y_1}\Big) = -\mathcal{U}_\epsilon\Big(\frac{\partial u}{\partial x}\Big)\frac{\partial X}{\partial y_1}\Big(\frac{x}{\epsilon^\alpha}\Big).$$

Hence, from iii) we get the desired convergences.

□

Remark 1.3.7. *Notice that the homogenized problem* (1.3.5) *is well posed and from standard elliptic regularity theory we can ensure that* $u \in H^2(0,1)$.

Note that, in case that we assume extra regularity conditions, for instance the function $g \in C^0(\mathbb{R},\mathbb{R})$, *we can define the function* X *as follows*

$$X(y_1) = \int_0^{y_1} 1 - \frac{1}{g\mathcal{M}(\frac{1}{g})}\, dy_1,$$

which belongs to $C^1(0,L)$ *and it is L-periodic. Then, using similar arguments as Remark 1.2.10 it follows that*

$$\lim_{\epsilon\to 0} \Big|\Big|\Big| u^\epsilon - u + \epsilon^\alpha \frac{\partial u}{\partial x} X^\epsilon \Big|\Big|\Big|_{H^1(R^\epsilon)} = 0,$$

where $X^\epsilon(x) \equiv X\Big(\dfrac{x}{\epsilon^\alpha}\Big)$, $\forall x \in (0,1)$.

1.4. Extremely high oscillatory behavior, $\alpha > 1$

In this section we analyze the behavior of the solutions of the Neumann problem (1.0.2) as the upper boundary of the thin domains presents a very high oscillatory boundary. Then, the thin domain is defined as follows

$$R^\epsilon = \Big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ 0 < y < \epsilon\, g(x/\epsilon^\alpha)\Big\}, \quad \alpha > 1. \tag{1.4.1}$$

We require that the function $g(\cdot)$ satisfies the hypothesis **(Hg)** from Section 1.1. Moreover we assume that $0 < g_0$ which implies that R^ϵ is connected.

Note that, since $\alpha > 1$ the order of the frequency of the oscillations is larger than the order of the height of the thin domain with respect to the parameter ϵ. Indeed, the ratio $\dfrac{\epsilon}{\epsilon^\alpha}$ tends to infinity.

We would like to point out that even though we use the unfolding operator to get the homogenized limit problem, the approach is different to the two previous cases.

The roughness is so strong that we can not obtain a compactness theorem using the same arguments as Theorem 1.2.3 or Theorem 1.3.1. In particular, defining an operator Z_ϵ analogous as the previous cases

$$Z_\epsilon := \frac{1}{\epsilon^\alpha}\Big(\mathcal{T}_\epsilon(\varphi^\epsilon) - \frac{1}{|Y^*|}\int_{Y^*} T_\epsilon(\varphi^\epsilon)\, dy_2 dy_1\Big),$$

does not help much to get a convergence result. Observe that in case $\alpha > 1$ the partial derivative respect to y_2, $\frac{\partial Z_\epsilon}{\partial y_2} = \epsilon^{1-\alpha}\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial y}\Big)$, is not bounded.

To overcome this difficulty we will divide the thin domain in two thin parts: one of them, R^ϵ_+, presents high oscillations and the other one, R^ϵ_-, is a non oscillating thin domain. Then, in order to get the convergence of the weak solutions of (1.0.2) and to identify the homogenized limit problem we apply the unfolding operator introduced in Definition 1.1.3 to the functions restricted to R^ϵ_+ and we introduce a rescaling operator to deal with the functions restricted to the thin non oscillating part.

Then, we consider these two open sets

$$\begin{aligned} R^\epsilon_+ &= \Big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ \epsilon g_0 < y < \epsilon\, g(x/\epsilon^\alpha)\Big\}, \\ R^\epsilon_- &= \Big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ 0 < y < \epsilon g_0\Big\}. \end{aligned}$$

Notice that

$$R^\epsilon = \mathrm{Int}\Big(\overline{R^\epsilon_+} \bigcup \overline{R^\epsilon_-}\Big).$$

Moreover, we set

$$\begin{aligned} Y^* &= \{(y_1,y_2) \in \mathbb{R}^2 \ :\ 0 < y_1 < L,\ 0 < y_2 < g(y_1)\}, \\ Y^*_+ &= \{(y_1,y_2) \in \mathbb{R}^2 \ :\ 0 < y_1 < L,\ g_0 < y_2 < g(y_1)\}, \\ R_- &= \big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ 0 < y < g_0\big\}, \\ R_+ &= \big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ g_0 < y < g_1\big\}. \end{aligned}$$

Remark 1.4.1. *Notice that the reference cell for the unfolding operator restricted to the oscillating part, Y^*_+, may be disconnected, see Figure 1.8.*

Figure 1.8: Disconnected reference cell Y^*_+

We first introduce an operator which allows us to rescale R^ϵ_- in order to work over a fixed domain.

1.4.1. Rescaling operator

Since R^ϵ_-, the lower part of the domain R^ϵ, has thickness ϵ we define an operator which allows us to interpret integrals over the thin domain R^ϵ_- as integrals over the fixed domain R_-. This operator is called rescaling operator and for any $\varphi \in L^2(R^\epsilon_-)$ it is defined as follows

$$\Pi_\epsilon(\varphi)(x,y) = \varphi(x,\epsilon y), \quad \forall (x,y) \in R_-. \tag{1.4.2}$$

Proposition 1.4.2. *The rescaling operator Π_ϵ has the following properties:*

i) Let $\varphi \in L^1(R^\epsilon_-)$. Then,

$$\int_{R_-} \Pi_\epsilon(\varphi)(x,y)\, dxdy = \frac{1}{\epsilon}\int_{R^\epsilon_-} \varphi(x,y)\, dxdy. \tag{1.4.3}$$

ii) Π_ϵ is linear and continuous from $\varphi \in L^p(R^\epsilon_-)$ to $\Pi_\epsilon(\varphi) \in L^p(R_-)$, $1 \le p \le \infty$. In addition, the following relationship exits between their norms

$$\|\Pi_\epsilon(\varphi)\|_{L^p(R_-)} = |||\varphi|||_{L^p(R^\epsilon_-)} \quad \text{for } 1 \le p < \infty,$$
$$\|\Pi_\epsilon(\varphi)\|_{L^\infty(R_-)} = |||\varphi|||_{L^\infty(R^\epsilon_-)}.$$

iii) For $\varphi \in W^{1,p}(R^\epsilon_-)$, $1 \le p \le \infty$ we have

$$\frac{\partial \Pi_\epsilon(\varphi)}{\partial x} = \Pi_\epsilon\Big(\frac{\partial \varphi}{\partial x}\Big), \qquad \frac{\partial \Pi_\epsilon(\varphi)}{\partial y} = \epsilon \Pi_\epsilon\Big(\frac{\partial \varphi}{\partial y}\Big).$$

iv) Let $\phi \in L^p(0,1)$, $1 \le p \le \infty$. Then, considering ϕ as a function defined in R^ϵ_- one has $\Pi_\epsilon(\phi) = \phi$.

Proof. These assertions are straightforward from the definition of the rescaling operator and their proofs are omitted. □

1.4.2. Homogenized limit problem

We obtain now the homogenization result based on the unfolding operator and the rescaling operator.

Theorem 1.4.3. *Let u^ϵ be the solution of problem (1.0.3) with $f^\epsilon \in L^2(R^\epsilon)$ satisfying $|||f^\epsilon|||_{L^2(R^\epsilon)} \le C$ for some positive constant C independent of $\epsilon > 0$. Assume the function $\hat{f}^\epsilon(x) = \frac{1}{\epsilon}\int_0^{\epsilon g(x/\epsilon^\alpha)} f^\epsilon(x,y)\, dy$ satisfies that there exists a function $\hat{f}$ such that*

$$\hat{f}^\epsilon \overset{\epsilon \to 0}{\rightharpoonup} \hat{f} \quad \textit{weakly in } L^2(0,1).$$

Then, there exists a unique element $u \in H^1(0,1)$ such that, as ϵ goes to zero,

$$\mathcal{T}_\epsilon(u^\epsilon) \stackrel{\epsilon\to 0}{\rightharpoonup} u \quad w - L^2\big((0,1); H^1(Y^*)\big),$$
$$\lim_{\epsilon\to 0} |||u^\epsilon - u|||_{L^2(R^\epsilon)} = 0.$$

Moreover, u is the unique weak solution of the following Neumann problem

$$\begin{cases} -g_0 u_{xx} + \dfrac{|Y^*|}{L} u = \hat{f}(x), \quad x \in (0,1), \\ \qquad u'(0) = u'(1) = 0. \end{cases} \tag{1.4.4}$$

Proof. Throughout this proof we denote by $\mathcal{T}_\epsilon$ the unfolding operator associated to the cell Y^*, $\mathcal{T}_\epsilon : L^2(R^\epsilon) \to L^2((0,1)\times Y^*)$, and by $\mathcal{T}_\epsilon^+$ the unfolding operator associated to the cell Y_+^*, $\mathcal{T}_\epsilon^+ : L^2(R_+^\epsilon) \to L^2((0,1)\times Y_+^*)$.

The first point is to obtain a uniform bound of the solutions u^ϵ. To do so, we choose u^ϵ as test function in (1.0.3). Then, we have

$$\Big\|\frac{\partial u^\epsilon}{\partial x}\Big\|^2_{L^2(R^\epsilon)} + \Big\|\frac{\partial u^\epsilon}{\partial y}\Big\|^2_{L^2(R^\epsilon)} + \|u^\epsilon\|^2_{L^2(R^\epsilon)} \le \|f^\epsilon\|_{L^2(R^\epsilon)} \|u^\epsilon\|_{L^2(R^\epsilon)}.$$

In view of this estimate and using that $|||f^\epsilon|||_{L^2(R^\epsilon)} \le C$, with C independent of the parameter ϵ, we easily get that

$$|||u^\epsilon|||_{H^1(R^\epsilon)} \le C \quad \forall \epsilon > 0. \tag{1.4.5}$$

From (1.4.5) and using Proposition 1.1.14 we have that there exists $u \in H^1(0,1)$ such that

$$\mathcal{T}_\epsilon(u^\epsilon) \stackrel{\epsilon\to 0}{\rightharpoonup} u \quad \mathrm{w} - L^2\big((0,1); H^1(Y^*)\big), \tag{1.4.6}$$

$$\lim_{\epsilon\to 0} |||u^\epsilon - u|||_{L^2(R^\epsilon)} = 0, \tag{1.4.7}$$

$$\mathcal{T}_\epsilon(u^\epsilon) \stackrel{\epsilon\to 0}{\longrightarrow} u \quad \mathrm{s} - L^2\big((0,1)\times Y^*\big).$$

In order to simplify the notation we denote the restriction of the solution to R_+^ϵ and R_-^ϵ as follows

$$u_+^\epsilon := u^\epsilon|_{R_+^\epsilon} \text{ and } u_-^\epsilon := u^\epsilon|_{R_-^\epsilon}.$$

From a priori estimate (1.4.5) and property vi) in Proposition 1.1.4 we have that $\mathcal{T}_\epsilon^+\Big(\dfrac{\partial u_+^\epsilon}{\partial x}\Big)$ is bounded and then, by weak compactness there exists a function $u_1 \in L^2\big((0,1)\times Y_+^*)\big)$ such that, up to subsequences,

$$\mathcal{T}_\epsilon\Big(\frac{\partial u_+^\epsilon}{\partial x}\Big) \stackrel{\epsilon\to 0}{\rightharpoonup} u_1 \quad \mathrm{w} - L^2\big((0,1)\times Y_+^*\big). \tag{1.4.8}$$

Now, we will prove that $u_1(x, y_1, y_2) = 0$ for a.e. $(x, y_1, y_2) \in (0,1)\times Y_+^*$.

To do this, we will define suitable test functions. Observe that since $g_0 = \min_{x\in\mathbb{R}}\{g(x)\}$ and $g(\cdot)$ is $L-$periodic there is, at least, a point $y_0 \in [0, L)$ where the minimum, g_0, is attained, that is, $g(y_0) = g_0$. Moreover, we have by definition that the segment

$$\{(y_0, y_2) \in \mathbb{R}^2 : y_2 \in (g_0, g_1)\} \cap Y_+^* = \emptyset. \tag{1.4.9}$$

Assume first that $y_0 > 0$, later on we deal with $y_0 = 0$. Then, for any $\phi \in \mathcal{D}(0, y_0)$ we define the following function

$$\psi(y_1) = \begin{cases} \displaystyle\int_0^{y_1} \phi(z)\, dz & \text{if } 0 \le y_1 < y_0, \\ 0 & \text{if } y_0 < y_1 < L. \end{cases} \tag{1.4.10}$$

Notice that ψ can be extended by $L-$periodicity and $\psi \in C^\infty[0, y_0) \cup C^\infty(y_0, L)$.

Then, we consider the following test function

$$\varphi^\epsilon(x, y) = \epsilon^\alpha \tilde{\varphi}\Big(x, \frac{y}{\epsilon}\Big)\psi\Big(\Big\{\frac{x}{\epsilon^\alpha}\Big\}_L\Big), \quad (x, y) \in R^\epsilon,$$

where $\varphi \in \mathcal{D}(R_+)$, ψ is defined in (1.4.10) and recall that $\tilde{}$ denotes the standard extension by zero. Note that, in view of (1.4.9) and definition of ψ, see (1.4.10), the functions φ^ϵ are continuous in R^ϵ.

Then, applying the unfolding operator introduced in Definition 1.1.3 to the restriction of φ^ϵ to the thin domain R_+^ϵ we get

$$\mathcal{T}_\epsilon^+(\varphi^\epsilon)(x, y_1, y_2) = \begin{cases} \epsilon^\alpha \varphi\Big(\epsilon^\alpha \Big[\frac{x}{\epsilon^\alpha}\Big]_L L + \epsilon^\alpha y_1, y_2\Big)\psi(y_1) & \text{for} \quad (x, y_1, y_2) \in I^\epsilon \times Y_+^*, \\ 0 & \text{for} \quad (x, y_1, y_2) \in \Lambda^\epsilon \times Y_+^*. \end{cases}$$

Moreover, by property vii) in Proposition 1.1.4 we have

$$\begin{aligned} \mathcal{T}_\epsilon^+\Big(\frac{\partial \varphi^\epsilon}{\partial x}\Big) &= \frac{1}{\epsilon^\alpha}\frac{\partial}{\partial y_1}\mathcal{T}_\epsilon^+(\varphi^\epsilon) = \epsilon^\alpha \mathcal{T}_\epsilon^+\Big(\frac{\partial \varphi}{\partial x}\Big)\psi(y_1) + \psi'(y_1)\mathcal{T}_\epsilon^+(\varphi), \\ \mathcal{T}_\epsilon^+\Big(\frac{\partial \varphi^\epsilon}{\partial y}\Big) &= \frac{1}{\epsilon}\frac{\partial}{\partial y_2}\mathcal{T}_\epsilon^+(\varphi^\epsilon) = \epsilon^{\alpha-1}\mathcal{T}_\epsilon^+\Big(\frac{\partial \varphi}{\partial y}\Big)\psi(y_1). \end{aligned}$$

Hence, since $\alpha > 1$ we get the following convergences

$$\begin{aligned} &\mathcal{T}_\epsilon^+(\varphi^\epsilon) \overset{\epsilon \to 0}{\longrightarrow} 0 \quad \mathrm{s} - L^2\big((0,1) \times Y_+^*\big), \\ &\mathcal{T}_\epsilon^+\Big(\frac{\partial \varphi^\epsilon}{\partial y}\Big) \overset{\epsilon \to 0}{\longrightarrow} 0 \quad \mathrm{s} - L^2\big((0,1) \times Y_+^*\big), \\ &\mathcal{T}_\epsilon^+\Big(\frac{\partial \varphi^\epsilon}{\partial x}\Big) \overset{\epsilon \to 0}{\longrightarrow} \psi'(y_1)\varphi(x, y_2) \quad \mathrm{s} - L^2\big((0,1) \times Y_+^*\big). \end{aligned} \tag{1.4.11}$$

Now, taking into account that φ^ϵ is the null function in R_-^ϵ we obtain the following integral equality from the weak formulation (1.0.3)

$$\int_{R_+^\epsilon} \Big\{\frac{\partial u_+^\epsilon}{\partial x}\frac{\partial \varphi^\epsilon}{\partial x} + \frac{\partial u_+^\epsilon}{\partial y}\frac{\partial \varphi^\epsilon}{\partial y} + u_+^\epsilon \varphi^\epsilon\Big\} dxdy = \int_{R_+^\epsilon} f^\epsilon \varphi^\epsilon dxdy.$$

Applying the unfolding operator, $\mathcal{T}_\epsilon^+$, and taking into account that $|||f^\epsilon|||_{L^2(R^\epsilon)} \le C$, Proposition 1.1.7 and convergences (1.4.6), (1.4.8) and (1.4.11) we get at the limit

$$\int_{(0,1)\times Y_+^*} u_1(x, y_1, y_2)\psi'(y_1)\varphi(x, y_2)\, dxdy_1dy_2 = 0.$$

This implies that for any $\varphi \in \mathcal{D}(R_+)$ and ψ defined as (1.4.10) we have

$$\int_{(0,1)\times(g_0,g_1)} \varphi(x, y_2)\Big[\int_{(0,L)} \tilde{u}_1(x, y_1, y_2)\psi'(y_1)\, dy_1\Big]\, dxdy_2 = 0.$$

Consequently, we get

$$\int_{(0,L)} \tilde{u}_1(x, y_1, y)\psi'(y_1)\, dy_1 = 0, \qquad \text{a.e. for } (x, y) \in R_+.$$

Then, from definition (1.4.10) we obtain that

$$\int_{(0,y_0)} \tilde{u}_1(x, y_1, y)\phi(y_1)\, dy_1 = 0, \quad \forall \phi \in \mathcal{D}(0, y_0) \text{ and a.e. for } (x, y) \in R_+.$$

Hence, we have that

$$\tilde{u}_1(x, y_1, y_2) = 0 \text{ for a.e. } (x, y_1, y_2) \in (0, 1) \times (0, y_0) \times (g_0, g_1). \tag{1.4.12}$$

Now, we repeat the same arguments defining ψ as follows

$$\psi(y_1) = \begin{cases} 0 & \text{if } 0 \le x < y_0, \\ \displaystyle\int_{y_0}^{y_1} \phi(z)\, dz - \int_{y_0}^{L} \phi(z)\, dz & \text{if } y_0 < x < L, \end{cases} \tag{1.4.13}$$

where $\phi \in \mathcal{D}(y_0, L)$. Notice that, ψ is $L-$periodic and $\psi \in C^\infty[0, y_0) \cup C^\infty(y_0, L)$.

Thus, using the same reasoning as above we get

$$\int_{(0,L)} \tilde{u}_1(x, y_1, y)\psi'(y_1)\, dy_1 = 0, \qquad \text{for a.e } (x, y) \in R_+.$$

Then, from definition (1.4.13) we obtain that

$$\int_{(q,L)} \tilde{u}_1(x, y_1, y)\phi(y_1)\, dy_1 = 0, \quad \forall \phi \in \mathcal{D}(y_0, L) \text{ and for a.e. } (x, y) \in R_+,$$

which implies that

$$\tilde{u}_1(x, y_1, y_2) = 0 \text{ for a.e } (x, y_1, y_2) \in (0, 1) \times (y_0, L) \times (g_0, g_1). \tag{1.4.14}$$

Hence, from (1.4.12) and (1.4.14) we can conclude that

$$u_1(x, y_1, y_2) = 0 \text{ for a.e. } (x, y_1, y_2) \in (0, 1) \times Y_+^*.$$

Finally, note that in case $y_0 = 0$ we may define

$$\psi(y_1) = \int_0^{y_1} \phi(z)\, dz \quad \text{if } 0 < x < L.$$

for any $\phi \in \mathcal{D}(0, L)$. Thus, taking into account that in this case we have

$$\{(0, y_2) \in \mathbb{R}^2 : y_2 \in (g_0, g_1)\} \cap Y_+^* = \emptyset, \text{ and } \{(L, y_2) \in \mathbb{R}^2 : y_2 \in (g_0, g_1)\} \cap Y_+^* = \emptyset,$$

we can ensure that the following test function is well defined

$$\varphi^\epsilon(x,y) = \epsilon^\alpha \tilde{\varphi}\Big(x, \frac{y}{\epsilon}\Big)\psi\Big(\Big\{\frac{x}{\epsilon^\alpha}\Big\}_L\Big), \quad (x,y) \in R^\epsilon.$$

Then, using exactly the same arguments as in the previous case we obtain $u_1 = 0$.

Therefore, we get

$$\mathcal{T}_\epsilon\Big(\frac{\partial u_+^\epsilon}{\partial x}\Big) \stackrel{\epsilon\to 0}{\rightharpoonup} 0 \quad \mathrm{w} - L^2\big((0,1)\times Y_+^*\big). \tag{1.4.15}$$

As far as the u_-^ϵ is concerned, from the a priori estimate (1.4.5) and taking into account properties ii) and iii) in Proposition 1.4.2 we know by weak compactness that there exists $u_- \in H^1(0,1)$ such that, up to subsequences,

$$\begin{aligned} &\Pi_\epsilon(u_-^\epsilon) \stackrel{\epsilon\to 0}{\rightharpoonup} u_- \ \mathrm{w} - H^1\big(R_-\big), \\ &\Pi_\epsilon\Big(\frac{\partial u_-^\epsilon}{\partial x}\Big) \stackrel{\epsilon\to 0}{\rightharpoonup} \frac{\partial u_-}{\partial x} \ \mathrm{w} - L^2\big(R_-\big). \end{aligned} \tag{1.4.16}$$

Moreover, from properties ii) and iv) of Proposition 1.4.2 we have

$$\|\Pi_\epsilon(u_-^\epsilon) - u\|_{L^p(R_-)} = |||u_-^\epsilon - u|||_{L^p(R_-^\epsilon)} \le |||u^\epsilon - u|||_{L^p(R^\epsilon)}.$$

Then, taking into account (1.4.7) we obtain

$$\Pi_\epsilon(u_-^\epsilon) \stackrel{\epsilon\to 0}{\rightharpoonup} u \quad \mathrm{s} - L^2\big(R_-\big),$$

which leads to $u(x) = u_-(x)$ for a.e. $x \in (0,1)$.

Finally, we obtain the limit weak formulation satisfied by u. Let us apply the unfolding and the rescaling operator to the original variational formulation (1.0.3). For $\phi \in H^1(0,1)$, we have

$$\begin{aligned} &\frac{1}{L}\int_{(0,1)\times Y_+^*} \mathcal{T}_\epsilon^+\Big(\frac{\partial u^\epsilon}{\partial x}\Big)\mathcal{T}_\epsilon^+\Big(\frac{\partial \phi}{\partial x}\Big)\,dxdy_1dy_2 + \frac{1}{L}\int_{(0,1)\times Y^*} \mathcal{T}_\epsilon(u^\epsilon)\mathcal{T}_\epsilon(\phi)dxdy_1dy_2 \\ &\quad + \frac{1}{\epsilon}\int_{R_{+1}^\epsilon} \frac{\partial u^\epsilon}{\partial x}\frac{\partial \phi}{\partial x}dxdy + \frac{1}{\epsilon}\int_{R_1^\epsilon} u^\epsilon\phi dxdy + \int_{R_-} \Pi_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big)\Pi_\epsilon\Big(\frac{\partial \phi}{\partial x}\Big)dxdy \\ &= \frac{1}{\epsilon}\int_{R^\epsilon} f^\epsilon \phi\, dxdy. \end{aligned}$$

Recall that R_{+1}^ϵ and R_1^ϵ are the subsets of R_+^ϵ and R^ϵ respectively which contain the corresponding part of the unique cell which is not totally included in the thin set, R_+^ϵ or R^ϵ.

Hence, taking into account the properties of the unfolding and the rescaling operator, the converges obtained above and the assumption on the function f^ϵ we can pass to the limit in the last equality, it leads to

$$\frac{1}{L}\int_{(0,1)\times Y^*} u\phi\, dxdy_1dy_2 + \int_{R_-} \frac{\partial u}{\partial x}\frac{\partial \phi}{\partial x}\, dxdy = \int_{(0,1)} \hat{f}\phi\, dx, \ \forall \phi \in H^1(0,1)$$

Consequently, we get that $u \in H^1(0,1)$ satisfies

$$\int_0^1 \Big\{ g_0 \frac{\partial u}{\partial x}\frac{\partial \phi}{\partial x} + \frac{|Y^*|}{L} u\phi \Big\}\, dx = \int_{(0,1)} \hat{f}\phi\, dx, \ \forall \phi \in H^1(0,1),$$

which is the variational formulation of (1.4.4).

□

Remark 1.4.4. *Observe that in case $f^\epsilon(x,y) = f(x)$ then*

$$\hat{f}^\epsilon(x) = g\Big(\frac{x}{\epsilon}\Big) f(x)$$

and from the average convergence for periodic functions (see, e.g., [52, p. xvi])

$$\hat{f}^\epsilon \overset{\epsilon\to 0}{\rightharpoonup} \hat{f} = f\mathcal{M}(g) = \frac{1}{L}\int_0^L g(y_1)\, dy_1\, f = \frac{|Y^*|}{L} f.$$

Hence the homogenized limit problem is given by

$$\begin{cases} -\dfrac{|Y^*_-|}{|Y^*|} u_{xx} + u = f, \quad x \in (0,1), \\ \qquad u'(0) = u'(1) = 0, \end{cases}$$

where $Y^*_- = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \in (0, L), 0 < y_2 < g_0\}$.

Note that we recover the homogenized limit problem obtained in [10] for Lipschitz domains.

To end this section we obtain some strong convergences for the sequence of the solutions which had not been obtained in previous papers. We show how the extremely oscillatory behavior affects the limit of the solutions. The roughness is so strong that the gradient of the solutions tends to zero in the upper part.

Proposition 1.4.5. *Under the same hypothesis as in Theorem 1.4.3 the solution satisfies the following convergences*

i) One has the following strong convergences

$$\mathcal{T}^+_\epsilon\Big(\frac{\partial u^\epsilon_+}{\partial x}\Big) \overset{\epsilon\to 0}{\longrightarrow} 0 \quad s - L^2\big((0,1)\times Y^*_+\big), \tag{1.4.17}$$

$$\mathcal{T}^+_\epsilon\Big(\frac{\partial u^\epsilon_+}{\partial y}\Big) \overset{\epsilon\to 0}{\longrightarrow} 0 \quad s - L^2\big((0,1)\times Y^*_+\big), \tag{1.4.18}$$

$$\Pi_\epsilon(u^\epsilon_-) \overset{\epsilon\to 0}{\longrightarrow} u \quad s - H^1\big(R_-\big),$$

$$\Pi_\epsilon\Big(\frac{\partial u^\epsilon_-}{\partial y}\Big) \overset{\epsilon\to 0}{\longrightarrow} 0 \quad s - L^2\big(R_-\big).$$

ii) $\displaystyle\lim_{\epsilon\to 0} \Big|\Big|\Big| \frac{\partial u^\epsilon}{\partial x} \Big|\Big|\Big|_{L^2(R^\epsilon_+)} = 0$ *and* $\displaystyle\lim_{\epsilon\to 0} \Big|\Big|\Big| \frac{\partial u^\epsilon}{\partial y} \Big|\Big|\Big|_{L^2(R^\epsilon_+)} = 0.$

Proof. i) To obtain the convergences we take $u^\epsilon - u$ as a test function in (1.0.3)

$$\int_{R^\epsilon} \Big\{\Big(\frac{\partial u^\epsilon}{\partial x}\Big)^2 - \frac{\partial u^\epsilon}{\partial x}\frac{\partial u}{\partial x} + \Big(\frac{\partial u^\epsilon}{\partial y}\Big)^2 + (u^\epsilon)^2 - u^\epsilon u\Big\} dxdy = \int_{R^\epsilon} f^\epsilon (u^\epsilon - u) dxdy. \tag{1.4.19}$$

Applying the unfolding and the rescaling operator we are allowed to pass to the limit in (1.4.19). Then, due to the convergences (1.4.15), (1.4.16) and the strong convergence (1.4.7) we get

$$\int_{(0,1)\times Y_+^*} \Big\{\mathcal{T}_\epsilon^+\Big(\frac{\partial u_+^\epsilon}{\partial x}\Big)^2 + \mathcal{T}_\epsilon^+\Big(\frac{\partial u_+^\epsilon}{\partial y}\Big)^2\Big\} dxdy_1dy_2 + \frac{1}{\epsilon}\int_{R_{+1}^\epsilon} |\nabla u^\epsilon|^2 dxdy$$
$$+ \int_{R_-} \Big\{\Pi_\epsilon\Big(\frac{\partial u_-^\epsilon}{\partial x}\Big)^2 - \Big(\frac{\partial u}{\partial x}\Big)^2 + \Pi_\epsilon\Big(\frac{\partial u_-^\epsilon}{\partial y}\Big)^2\Big\} dxdy \overset{\epsilon\to 0}{\longrightarrow} 0.$$

Therefore, by weak lower-semicontinuity we have

$$\begin{aligned} 0 &\le \liminf_{\epsilon\to 0} \int_{R_-} \Big\{\Pi_\epsilon\Big(\frac{\partial u_-^\epsilon}{\partial x}\Big)^2 - \Big(\frac{\partial u}{\partial x}\Big)^2\Big\} dxdy \\ &\le \limsup_{\epsilon\to 0} \int_{R_-} \Big\{\Pi_\epsilon\Big(\frac{\partial u_-^\epsilon}{\partial x}\Big)^2 - \Big(\frac{\partial u}{\partial x}\Big)^2\Big\} dxdy \\ &\le \Big\{\lim_{\epsilon\to 0} \int_{R_-} \Big\{\Pi_\epsilon\Big(\frac{\partial u_-^\epsilon}{\partial x}\Big)^2 - \Big(\frac{\partial u}{\partial x}\Big)^2\Big\} dxdy + \int_{R_-} \Pi_\epsilon\Big(\frac{\partial u_-^\epsilon}{\partial y}\Big)^2 dxdy \\ &+ \int_{(0,1)\times Y_+^*} \Big\{\mathcal{T}_\epsilon^+\Big(\frac{\partial u_+^\epsilon}{\partial x}\Big)^2 + \mathcal{T}_\epsilon^+\Big(\frac{\partial u_+^\epsilon}{\partial y}\Big)^2\Big\} dxdy_1dy_2 + \frac{1}{\epsilon}\int_{R_{+1}^\epsilon} |\nabla u^\epsilon|^2 dxdy\Big\} = 0. \end{aligned}$$

Consequently, we get

$$\begin{aligned} &\lim_{\epsilon\to 0} \int_{R_-} \Big\{\Pi_\epsilon\Big(\frac{\partial u_-^\epsilon}{\partial x}\Big)^2 - \Big(\frac{\partial u}{\partial x}\Big)^2\Big\} dxdy = 0 \\ &\lim_{\epsilon\to 0} \int_{R_-} \Pi_\epsilon\Big(\frac{\partial u_-^\epsilon}{\partial y}\Big)^2 dxdy = 0 \\ &\lim_{\epsilon\to 0} \int_{(0,1)\times Y_+^*} \Big\{\mathcal{T}_\epsilon^+\Big(\frac{\partial u_+^\epsilon}{\partial x}\Big)^2 + \mathcal{T}_\epsilon^+\Big(\frac{\partial u_+^\epsilon}{\partial y}\Big)^2\Big\} dxdy_1dy_2 = 0, \\ &\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_{R_{+1}^\epsilon} |\nabla u^\epsilon|^2 dxdy = 0. \end{aligned} \tag{1.4.20}$$

The desired convergences are then straightforward. Note that, due to weak convergence (1.4.16) we obtain by the Radon-Riesz property the following convergence

$$\Pi_\epsilon(u_-^\epsilon) \overset{\epsilon\to 0}{\longrightarrow} u \text{ s} - H^1(R_-).$$

ii) From convergences (1.4.17), (1.4.18), (1.4.20) and property v) of Proposition 1.1.16 one immediately has the convergences.

□

Chapter 2

Locally periodic thin domains with varying period

In the previous Chapter we have analyzed the behavior of solutions of the Poisson problem in a "purely periodic" thin domain, that is, a domain where the function g is periodic. For this we have used the unfolding operator method adapted to periodic thin domains.

In this chapter we want to go a little further and adapt the method to a non purely periodic thin domain. As a matter of fact, we analyze the behavior of solutions of the Poisson equation with homogeneous Neumann boundary conditions in a two-dimensional thin domain which presents "locally periodic" oscillations at the boundary. The oscillations are such that both the amplitude and period of the oscillations may vary in space. Continuing the study of the previous chapter we obtain the homogenized limit problem and a corrector result by extending the unfolding operator method as described in Chapter 1, to the case of locally periodic media.

We consider the Neumann problem for the Laplace operator

$$\begin{cases} -\Delta u^\epsilon + u^\epsilon = f^\epsilon & \text{in } R^\epsilon, \\ \dfrac{\partial u^\epsilon}{\partial \nu^\epsilon} = 0 & \text{on } \partial R^\epsilon, \end{cases} \tag{2.0.1}$$

where $f^\epsilon \in L^2(R^\epsilon)$, $\nu^\epsilon = (\nu_1^\epsilon, \nu_2^\epsilon)$ is the unit outward normal to ∂R^ϵ and $\frac{\partial}{\partial \nu^\epsilon}$ is the outside normal derivative. In this case, the domain R^ϵ is a two dimensional thin domain which presents a "resonant" oscillatory behavior at the top boundary, given by

$$R^\epsilon = \{(x, y) \in \mathbb{R}^2 \mid x \in (0, 1),\ 0 < y < \epsilon\, G(x, x/\epsilon)\} \tag{2.0.2}$$

where the smooth function G, defined as

$$\begin{aligned} G : (0,1) \times \mathbb{R} &\longrightarrow (0, +\infty) \\ (x, y) &\longrightarrow G(x, y) \end{aligned}$$

satisfies that there exist two positive constants G_0, G_1 with

$$0 < G_0 \leq G(x, y) \leq G_1, \quad \forall (x, y) \in (0, 1) \times \mathbb{R}.$$

Moreover, for each $x \in (0,1)$, the function $G(x,\cdot)$ is $l(x)$-periodic, with the function $l(\cdot)$ not being necessarily constant. This constitutes the main novelty of this work: we consider domains where both the amplitude and frequency of the oscillations depend on x (see Figure 2.1). In this respect we are deviating from the purely periodic case, which is the most common setting in homogenization theory and we are interested in analyzing how the geometry of the domain, the varying amplitude and period of the function G, affects the homogenized limit problem.

Observe that our setting includes the case of purely periodic oscillations, that is $G(x,y) = G(y)$ and the case where the thin domain is locally periodic with constant period, for instance, $G(x, x/\epsilon) = a(x) + b(x)g(x/\epsilon)$ where $a, b : (0,1) \to \mathbb{R}$ are $\mathcal{C}^1$ functions and $g : \mathbb{R} \to \mathbb{R}$ is an L-periodic smooth function (see for instance [9]). But, it also includes the very important and relevant case where the period changes as we vary x. For instance, $G(x, x/\epsilon) = a(x) + b(x)g(l(x)x/\epsilon)$ where $l : \mathbb{R} \to \mathbb{R}$ is certain smooth function.

Figure 2.1: Thin domain R^ϵ with a locally periodic oscillatory boundary

As it is shown in Chapter 1, see Section 1.2, the purely periodic case can be addressed by applying the unfolding periodic method. Recall that, if $G : [0, L] \to \mathbb{R}$ is the $L-$periodic function which defines the oscillatory boundary and the representative cell is given by

$$Y^* = \{(y_1, y_2) \in \mathbb{R}^2 \ : \ 0 < y_1 < L, \quad 0 < y_2 < G(y_1)\}$$

then the limit equation is shown to be

$$\begin{cases} -q_0 w_{xx} + w = f(x), \quad x \in (0,1) \\ w'(0) = w'(1) = 0 \end{cases} \tag{2.0.3}$$

where

$$q_0 = \frac{1}{|Y^*|} \int_{Y^*} \Big\{ 1 - \frac{\partial X}{\partial y_1}(y_1, y_2) \Big\} dy_1 dy_2,$$

and X is the unique solution of certain PDE problem posed in the basic cell Y^*. Also, the case where the function $G(x,\cdot)$ is L-periodic with L independent of x, that is, locally periodic case with fixed period, was analyzed in [9]. With the method of oscillatory test functions applied first to the case of piecewise periodic case and then with a domain perturbation argument, the following limit problem was obtained:

$$\begin{cases} -\dfrac{1}{p(x)}(q(x)w_x)_x + w = f(x), \quad x \in (0,1) \\ w'(0) = w'(1) = 0 \end{cases} \tag{2.0.4}$$

where

$$q(x) = \int_{Y^*(x)} \Big\{1 - \frac{\partial X(x, y_1, y_2)}{\partial y_1}\Big\} dy_1 dy_2, \tag{2.0.5}$$

$$p(x) = |Y^*(x)| \tag{2.0.6}$$

and the function $(y_1, y_2) \to X(x, y_1, y_2)$ is the unique solution of an appropriate PDE problem posed in the basic cell $Y^*(x)$ which depends on the variable x and it is given by

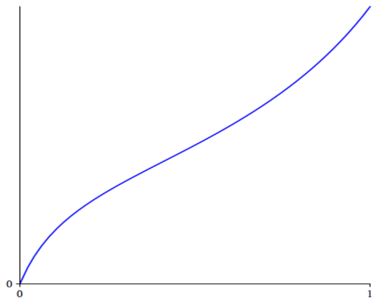
$$Y^*(x) = \{(y_1, y_2) \in \mathbb{R}^2 \ : \ 0 < y_1 < L, \quad 0 < y_2 < G(x, y_1)\}.$$

Notice that if we assume that $Y^*(x) \equiv Y^*$, then we recover the homogenized problem in the purely periodic case.

The analysis performed in [9] uses in a very definite way that the period of the function $G(x, \cdot)$ is independent of x and the techniques employed do not apply in a very straightforward way to the more general case of a varying period.

When dealing with a thin domain R^ϵ as defined in (2.0.2), where the function $G(x, \cdot)$ is periodic of period $l(x)$, we may distinguish two different situations.

On one hand, if the function $h(x) = \frac{x}{l(x)}$ satisfies $h'(x) > 0$ for all $x \in [0, 1]$ then $h : (0, 1) \to \Big(0, \frac{1}{l(1)}\Big)$ is a diffeomorphism, see Figure 2.2.



(a) Function $h(x) = \frac{x}{l(x)}$

(b) Corresponding thin domain

Figure 2.2: $h(\cdot)$ is a diffeomorphism

In this particular case, if seems reasonable to perform the following change of variables

$$\begin{aligned} Z^\epsilon : R^\epsilon \quad &\longrightarrow R_1^\epsilon \\ (x, y) \quad &\longrightarrow (x_1, x_2) := \big(h(x), y\big) = \Big(\tfrac{x}{l(x)}, y\Big). \end{aligned}$$

which transforms the thin domain R^ϵ into another thin domain, R_1^ϵ, having an oscillatory boundary given by the $1-$periodic function $H(x, y) = G(h^{-1}(x), l(h^{-1}(x))y)$, that is

$$R_1^\epsilon = \Big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0, 1/l(1)), \ 0 < x_2 < \epsilon\, H(x_1, x_1/\epsilon)\Big\}.$$

In this new system of coordinates, problem (2.0.1) is transformed into

$$\begin{cases} -h'(h^{-1}(x_1))\mathrm{div}\big(B(v^\epsilon)\big) + v^\epsilon = f^\epsilon \quad \text{in } R_1^\epsilon, \\ B(v^\epsilon)\cdot\eta = 0 \quad \text{on } \partial R_1^\epsilon, \\ v^\epsilon = u^\epsilon \circ (Z^\epsilon)^{-1} \text{ in } R_1^\epsilon \end{cases} \tag{2.0.7}$$

where η denotes the unit outward normal vector field to ∂R_1^ϵ and

$$B(v) = \Big(h'(h^{-1}(x_1))\frac{\partial v}{\partial x_1}, \frac{1}{h'(h^{-1}(x_1))}\frac{\partial v}{\partial x_2}\Big).$$

Under this change of variables, we have transformed the thin domain with variable amplitude and period of the oscillations into a locally periodic thin domain with constant period, although the amplitude continues to vary with space (that is, the function $H(x,y)$ actually depends on x). Moreover, in this case, the transformed differential operator, see (2.0.7), is now with non-constant coefficients. Now, we may proceed using the techniques from [9] or try to use the unfolding operator method adapted to the situation of variable amplitude, which is a particular case of what we are developing in this paper.

On the other hand, if the function $h(x) = \frac{x}{l(x)}$ is not a diffeomorphism, we cannot perform this change of variables. With the techniques we develop in this paper we will be able to address this situation. As a matter of fact we will assume the following hypothesis

(H) The function $l(\cdot)$ is $\mathcal{C}^1$ and there exist positive constants l_0, l_1 such that $0 < l_0 \leqslant l(x) < l_1$. Moreover, for all $k \in \mathbb{R}$ the points $x \in (0,1)$ such that $x = kl(x)$ is a finite set and if $A = \{x \in (0,1) : xl'(x) = l(x)\}$, then $\mu\{A\} = 0$, where μ denotes the Lebesgue measure.

This hypothesis contemplates the possibility that h has a finite number of maxima and minima (see for instance Figure 2.4) or even the more degenerate situation where the function $h(x)$ has an infinite number of critical points (see Figure 2.5). For both cases, it does not seem possible to perform a change of variables that transform the problem in a domain with fixed period.

As a matter of fact the function G which describes the oscillatory boundary of the thin domains of Figure 2.4 and 2.5 is given by $G(x,y) = b(x) + \cos\left(\frac{2\pi y}{l(x)}\right)$, where $b(\cdot)$ is a smooth positive function with the following shape

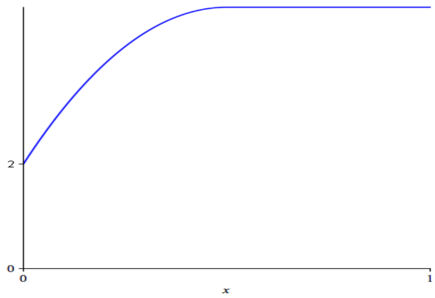
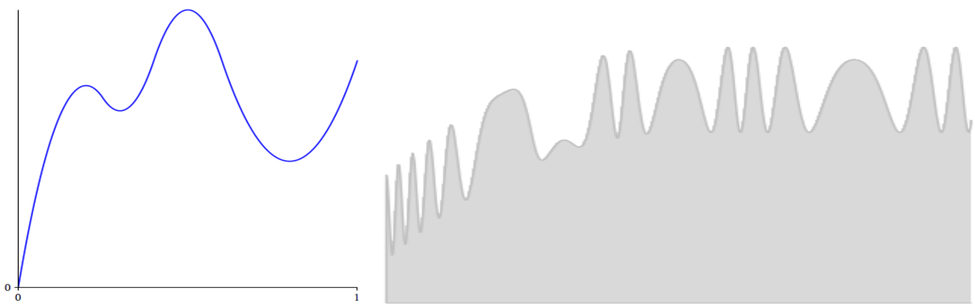


Figure 2.3: Function $b(\cdot)$

Then, for both Figure 2.4 and Figure 2.5 we have $G(x, x/\epsilon) = b(x) + \cos\left(\frac{2\pi h(x)}{\epsilon}\right)$, where $h(\cdot)$ is depicted in the figures.[1]



(a) Function $h(x) = \frac{x}{l(x)}$ (b) Corresponding thin domain

Figure 2.4: A is a finite set

Our proposal consist in avoiding to make a change of variables and rather adapt the Unfolding Operator method, which was initially devised to tackle purely periodic problems, to this new "locally periodic" situation.

Let us mention that this adjustment of the method applies also to the case where the period does not depend on the spatial variable x and we may consider this as a different method to obtain the results from [9]. Moreover, the possibility to apply the unfolding operator method to a non-periodic situation express the robustness and power of the method.

Our results, which were announced in [13] for the simpler and more intuitive case where $h'(\cdot) > 0$, will allow us to obtain the homogenized limit problem, together

[1] The actual analytic definition of $h(\cdot)$ is

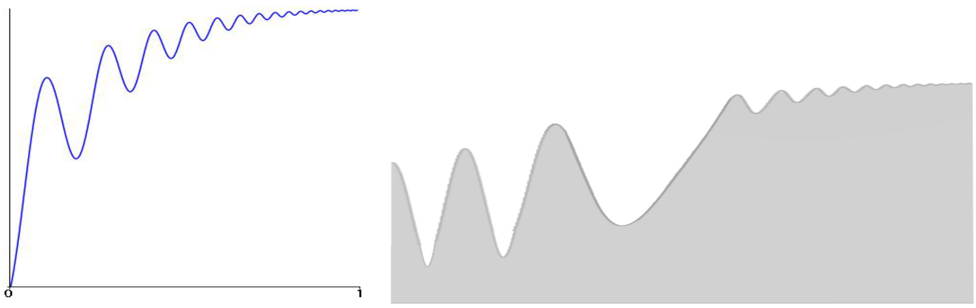
Figure 2.4

$$h(x) = \begin{cases} -100x^2 + 40x & \text{if } 0 < x < 0.25, \\ 100x^2 - 60x + 12,5 & \text{if } 0.25 \le x < 0.4, \\ -100x^2 + 100x - 19,5 & \text{if } 0.4 \le x < 0.6, \\ 50x^2 - 80x + 34,5 & \text{if } 0.6 \le x < 1, \end{cases}$$

Figure 2.5

$$h(x) = \begin{cases} ax & \text{if } 0 < x \le 0.2, \\ c - (1-x)^4\left(2 + sin\left(\frac{1}{1-x}\right)\right) & \text{if } 0.2 < x < 1, \end{cases}$$

Constants a and c were calculated in order to get $h \in C^1(0,1)$. Approximately $a = 5,837714188$ and $c = 2,375446938$.



(a) Function $h(x) = \frac{x}{l(x)}$ (b) Corresponding thin domain

Figure 2.5: A is a infinite numerable set

with a corrector result, for problems defined on thin domains with locally-periodic oscillatory boundary.

Summarizing our results, we obtain (see Theorem 2.4.1) the following limit problem:

$$\begin{cases} -\dfrac{l(x)}{|Y^*(x)|}\big(r(x)u_x\big)_x + u = f, \quad x \in (0,1) \\ u'(0) = u'(1) = 0 \end{cases}$$

with

$$r(x) = \frac{1}{l(x)} \int_{Y^*(x)} \Big\{1 - \frac{\partial X(x, y_1, y_2)}{\partial y_1}(y_1, y_2)\Big\} dy_1 dy_2$$

where for fixed $x \in (0,1)$, $Y^*(x)$ is the basic cell at x, given by

$$Y^*(x) = \{(y_1, y_2) \in \mathbb{R}^2 \ : \ 0 < y_1 < l(x), \quad 0 < y_2 < G(x, y_1)\} \tag{2.0.8}$$

and the function $(y_1, y_2) \to X(x, y_1, y_2)$, which is defined for $(y_1, y_2) \in Y^*(x)$, is the unique solution which is $l(x)$-periodic in the variable y_1 of the following auxiliary problem posed in the basic cell $Y^*(x)$

$$\begin{cases} -\Delta_{(y_1,y_2)} X(x, y_1, y_2) = 0 \text{ in } Y^*(x), \\ \dfrac{\partial X(x, \cdot, \cdot)}{\partial N(x)} = 0 \text{ on } B_2(x), \\ \dfrac{\partial X(x, \cdot, \cdot)}{\partial N(x)} = N_1(x) \text{ on } B_1(x), \\ \displaystyle\int_{Y^*(x)} X(x, y_1, y_2)\, dy_1 dy_2 = 0, \end{cases}$$

where $B_1(x)$ and $B_2(x)$ are the upper and lower boundary of $Y^*(x)$ respectively, that is,

$$\begin{aligned} B_1(x) &= \{(y_1, G(x, y_1)) : 0 < y_1 < l(x)\}, \\ B_2(x) &= \{(y_1, 0) : 0 < y_1 < l(x)\} \end{aligned} \tag{2.0.9}$$

and $N(x) = (N_1(x), N_2(x))$ is the outward unit normal at $\partial Y^*(x)$, which obviously depend on $x \in (0,1)$ and in particular it is given by

$$\begin{aligned} &N(x) = (0,-1) \qquad \text{at } B_2(x), \\ &N(x) = \Big(\frac{-G_y(x,y_1)}{\sqrt{1+(G_y(x,y_1))^2}}, \frac{1}{\sqrt{1+(G_y(x,y_1))^2}} \Big) \qquad \text{at } B_1(x), \end{aligned} \tag{2.0.10}$$

with $G_y = \dfrac{\partial G}{\partial y}$.

Notice that if $l(x)$ is a constant independent of $x \in (0,1)$ we recover (2.0.4), (2.0.5), (2.0.6).

In addition, adding the suitable corrector function we get the following corrector result (see Theorem 2.5.3)

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^{1/2}} \Big|\Big| u^\epsilon - u + \epsilon \frac{\partial u}{\partial x} X^\epsilon \Big|\Big|_{H^1(R^\epsilon)} = 0,$$

where $X^\epsilon(x,y) \equiv X(x, x/\epsilon, y/\epsilon)$ for all $(x,y) \in R^\epsilon$.

In this chapter we give a comprehensive presentation of the unfolding method for thin domains with a locally periodic oscillatory boundary emphasizing the similarities and the main differences respect to the periodic unfolding operator introduced in the previous chapter. We believe that our extension of the unfolding operator method to a locally periodic setting with varying period may help to get a better understanding of how to deal with other related problems in different situations.

The chapter is organized as follows.

- In Section 2.1, we fix some notation that will be used throughout the chapter.
- In Section 2.2 we construct the unfolding operator for locally periodic thin domains and show some important properties.
- In Section 2.3, we present some convergence results related to the unfolding operator. The essential result for the work is the compactness Theorem 2.3.9.
- In Section 2.4, we apply the previous results to obtain the homogenized limit problem. The Theorem 2.4.1 shows two equivalent formulations of the homogenized limit problem.
- Finally, in Section 2.5, we introduce an averaging operator $\mathcal{U}_\epsilon$, the adjoint of the unfolding operator, and we prove its main properties. To end this section, we use the averaging operator to obtain a corrector result.

Remark 2.0.6. *Part of the results of this chapter are announced in [13] and are published in [15].*

2.1. Some more notation

We include in this section some more notation which is complementary to the Notation Section at the beginning of the thesis. The notation included here is more specific for this chapter.

i) Observe that properly speaking there is not a basic cell associated to the domain R^ϵ, since the periodicity properties vary from point to point in $x \in (0,1)$. Nevertheless, we will refer to the set $Y^*(x)$ defined in (2.0.8) as the basic cell at x. Notice that all these "basic cells" satisfy $Y^*(x) \subset Y^*$ where $Y^* = (0, l_1) \times (0, G_1)$. Moreover, we will maintain the notation of $B_1(x)$ and $B_2(x)$ as the upper and lower boundary of $Y^*(x)$ and $N(x)$ as the outward normal at $Y^*(x)$, see (2.0.9) and (2.0.10).

ii) As it is shown in the previous chapter, the basic idea of the unfolding operator method in a purely periodic setting is to transform functions defined in R^ϵ into functions defined in a fixed domain. With analogy to this case, we will consider the domain

$$W = \{(x, y_1, y_2) \in \mathbb{R}^3 \, : \, x \in (0,1), \; (y_1, y_2) \in Y^*(x)\}$$

and $\hat{W} = \{(x, y_1, y_2) \in \mathbb{R}^3 \, : \, x \in (0,1), \; y_1 \in \mathbb{R} \text{ and } 0 < y_2 < G(x, y_1)\}$. Notice that $W = \{(x, y_1, y_2) \in \hat{W} : 0 < y_1 < l(x)\}$ and $W \subset (0,1) \times Y^*$. Moreover, W_0 denotes the bottom boundary of W, that is

$$W_0 = \{(x, y_1, 0) \in \mathbb{R}^3 : \, x \in (0,1), \; y_1 \in (0, l(x)\}.$$

iii) Given an interval $(a,b) \subset \mathbb{R}$ we denote its characteristic function by $\chi_{(a,b)}$. χ^ϵ is the characteristic function of R^ϵ and χ the characteristic function of W. Moreover, as we have mentioned in the Notation Section, $\tilde{}$ is the standard extension by zero operator.

iv) We will use the subindex # to denote periodicity with respect to the y_1 variable in the following sense. For every fixed $x \in (0,1)$, the space $C^k_\#\big(Y^*(x)\big)$ consists of all functions φ which are obtained as restrictions to $Y^*(x)$ of functions in $C^k(\mathbb{R}^2)$ which are $l(x)$-periodic in the first variable. That is

$$C^k_\#\big(Y^*(x)\big) = \{\varphi_{|Y^*(x)} : \varphi \in C^k(\mathbb{R}^2), \, \varphi(y_1 + l(x), y_2) = \varphi(y_1, y_2), \quad \forall (y_1, y_2) \in \mathbb{R}^2\}.$$

This is a Banach space with the usual norm $\|\cdot\|_{C^k(Y^*(x))}$.

Also, $W^{1,p}_\#\big(Y^*(x)\big)$ is the completion of $C^\infty_\#\big(Y^*(x)\big)$ for the norm $W^{1,p}\big(Y^*(x)\big)$. Moreover, the space $C^\infty_\#(W)$ is the space of functions $\varphi \in C^\infty(\mathbb{R}^3)$ restricted to W which are periodic in y_1 of period $l(x)$ for fixed $x \in (0,1)$, that is: $\varphi(x, y_1 + l(x), y_2) = \varphi(x, y_1, y_2)$.

v) Consider also the spaces of Banach space-valued functions $L^p\Big((0,1); W^{1,q}_\#\big(Y^*(x)\big)\Big)$, $L^p\Big((0,1); C^k_\#\big(Y^*(x)\big)\Big)$ and $W^{1,p}\Big((0,1); C^k_\#\big(Y^*(x)\big)\Big)$ in the following way:

(1) The space $L^p\Big((0,1);W^{1,q}_{\#}\big(Y^*(x)\big)\Big)$ consists of the measurable functions $\varphi : W \to \mathbb{R}$ such that $\varphi(x) \in W^{1,q}_{\#}\big(Y^*(x)\big)$ a.e. $x \in (0,1)$, with

$$\|\varphi\|_{L^p\big((0,1);W^{1,q}_{\#}\big(Y^*(x)\big)\big)} \equiv \begin{cases} \Big(\displaystyle\int_0^1 \|\varphi(x)\|^p_{W^{1,q}_{\#}\big(Y^*(x)\big)}\, dx\Big)^{1/p} < \infty,\ 1 \le p < \infty \\ \displaystyle\operatorname*{ess\,sup}_{x\in(0,1)} \|\varphi(x)\|_{W^{1,q}_{\#}\big(Y^*(x)\big)} < \infty, \quad p = \infty. \end{cases}$$

Notice that $L^2\Big((0,1);H^1_{\#}\big(Y^*(x)\big)\Big)$ actually coincides with the space of functions $\varphi \in L^2(W)$ such that $\frac{\partial\varphi}{\partial y_1}$, $\frac{\partial\varphi}{\partial y_2}$ belong to $L^2(W)$ and $\varphi(x,\cdot,\cdot)$ is $l(x)$-periodic in the first variable y_1.

(2) The space $L^p\Big((0,1);C^k_{\#}\big(Y^*(x)\big)\Big)$ comprises all strongly measurable functions $\varphi : W \to \mathbb{R}$ such that $\varphi(x) \in C^k_{\#}\big(\, Y^*(x)\,\big)$ and

$$\|\varphi\|_{L^p\Big((0,1);C^k_{\#}\big(\,Y^*(x)\,\big)\Big)} \equiv \begin{cases} \Big(\displaystyle\int_0^1 \|\varphi(x)\|^p_{C^k_{\#}\big(Y^*(x)\big)}\, dx\Big)^{1/p} < \infty,\ 1 \le p < \infty \\ \displaystyle\operatorname*{ess\,sup}_{x\in(0,1)} \|\varphi(x)\|_{C^k_{\#}\big(Y^*(x)\big)} < \infty, \quad p = \infty. \end{cases}$$

(3) The Banach space $W^{1,p}\Big((0,1);C^k_{\#}\big(\,Y^*(x)\,\big)\Big)$ consists of all functions $\varphi \in L^p\Big((0,1);C^k_{\#}\big(\,Y^*(x)\,\big)\Big)$ such that $\frac{\partial\varphi}{\partial x}$ exists in the weak sense and belongs to $L^p\Big((0,1);C^k_{\#}\big(\,Y^*(x)\,\big)\Big)$. Furthermore, for $1 \le p < \infty$

$$\|\varphi\|_{W^{1,p}\Big((0,1);C^k_{\#}\big(\,Y^*(x)\,\big)\Big)} \equiv \Big(\int_0^1 \|\varphi(x)\|^p_{C^k_{\#}\big(Y^*(x)\big)} + \Big\|\frac{\partial\varphi}{\partial x}\Big\|^p_{C^k_{\#}\big(Y^*(x)\big)}\, dx\Big)^{1/p},$$

and for $p = \infty$

$$\|\varphi\|_{W^{1,\infty}\Big((0,1);C^k_{\#}\big(\,Y^*(x)\,\big)\Big)} \equiv \operatorname*{ess\,sup}_{x\in(0,1)} \Big(\|\varphi(x)\|_{C^k_{\#}\big(Y^*(x)\big)} + \Big\|\frac{\partial\varphi}{\partial x}\Big\|_{C^k_{\#}\big(Y^*(x)\big)}\Big).$$

We usually write $H^1\Big((0,1);C^k_{\#}\big(\,Y^*(x)\,\big)\Big) = W^{1,2}\Big((0,1);C^k_{\#}\big(\,Y^*(x)\,\big)\Big)$.

2.2. The unfolding operator in a thin domain with locally periodic oscillatory boundary

In this section we construct the unfolding operator for the locally periodic case and show some basic properties. Due to the lack of periodicity, a delicate point in this construction is to define an appropriate partition of the limit segment $I = (0,1)$ which will be in accordance with the oscillatory behavior of the thin domain

(2.0.2). In periodic homogenization one can always choose the partition given by the constant period L, $\{0, \epsilon L, 2\epsilon L, \cdots, \epsilon N^\epsilon L, 1\}$. However, it is dramatically different in the presence of a variable period.

Then, we will start defining the concept of "admissible partition" of the interval $(0,1)$.

Definition 2.2.1. *An admissible partition is given by the family of ordered numbers* $\{x_k^\epsilon\}_{k=0}^{N_\epsilon+1}$ *for all* $0 < \epsilon \leq \epsilon_0$*, satisfying*

$$0 = x_0^\epsilon < x_1^\epsilon < \ldots < x_{N_\epsilon}^\epsilon < x_{N_\epsilon+1}^\epsilon = 1$$

Moreover, for almost every point $x \in (0,1)$ *there exist* $0 < \epsilon_1 \leq \epsilon_0$ *and a constant* $C = C(x)$ *such that for all* $0 < \epsilon < \epsilon_1 \leq \epsilon_0$ *there is a point* x_k^ϵ *of the partition which satisfies* $|x - x_k^\epsilon| \leq C\epsilon$.

We will refer to the partition as $\{x_k^\epsilon\}$.

Remark 2.2.2. *Notice that we do not require that the distance between any two consecutive points of the partition is uniformly bounded. Indeed, the points of an admissible partition do not necessarily satisfy that there exists* C *independent of* ϵ *such that*

$$|x_{k+1}^\epsilon - x_k^\epsilon| \leq C\epsilon, \ \forall k \in \{0, \ldots, N_\epsilon\}.$$

See Definition 2.3.1 as an example of admissible partition which, in general, does not satisfy the estimate above, see Figure 2.7.

The results of this section will be proved for a general admissible partition $\{x_k^\epsilon\}$.

Observe that, for every $x \in (0,1)$ there exists a unique element of the partition, x_k^ϵ, such that $x \in [x_k^\epsilon, x_{k+1}^\epsilon)$. By the analogy to the periodic case introduced in the previous chapter, we denote this point x_k^ϵ by $[x]_\epsilon$. In addition, since the partition is not equally spaced we consider for every $x \in (x_k^\epsilon, x_{k+1}^\epsilon)$ the factor $\Gamma_{[x]_\epsilon}$ given by

$$\Gamma_{[x]_\epsilon} := \frac{x_{k+1}^\epsilon - x_k^\epsilon}{l(x_k^\epsilon)}.$$

Remark 2.2.3. *Notice that the factor* $\Gamma_{[x]_\epsilon}$ *defines a linear function which transforms the interval* $(0, l(x_k^\epsilon))$ *into the interval* $(0, x_{k+1}^\epsilon - x_k^\epsilon)$

$$\begin{aligned} l_{[x]_\epsilon} : (0, l(x_k^\epsilon)) &\longrightarrow (0, x_{k+1}^\epsilon - x_k^\epsilon) \\ y_1 &\longrightarrow \Gamma_{[x]_\epsilon} y_1. \end{aligned}$$

Then, for each $x \in (0,1)$ there is a unique point in $y_1 \in \big(0, l([x]_\epsilon)\big)$ such that

$$x = [x]_\epsilon + \Gamma_{[x]_\epsilon} y_1.$$

Using this change of scale we are in a position to define the Unfolding Operator in our setting.

Definition 2.2.4. *Let $\{x_k^\epsilon\}$ be a general admissible partition as in Definition 2.2.1. Let φ be a Lebesgue-measurable function in R^ϵ. We define the unfolding operator $\mathcal{T}_\epsilon$ associated to the partition $\{x_k^\epsilon\}$, acting on φ, as the function $\mathcal{T}_\epsilon(\varphi)$ defined in $(0,1)\times Y^*$ as:*

$$\mathcal{T}_\epsilon(\varphi)(x,y_1,y_2) = \begin{cases} \widetilde{\varphi}\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \epsilon y_2\Big) & \textit{for} \quad y_1 \in \big(0, l([x]_\epsilon)\big), \\ 0 & \textit{for} \quad y_1 \in \big(l([x]_\epsilon), l_1\big). \end{cases}$$

As in classical periodic homogenization, the unfolding operator reflects two scales: the "macroscopic" scale x gives the position in the interval $(0,1)$ and the "microscopic" scale (y_1,y_2) gives the position in the cell Y^*. However, due to the locally periodic oscillations of the domain R^ϵ, the definition given here differs from the introduced in periodic cases. In this case, we do not have a fixed cell that describes the domain R^ϵ, therefore, in the definition we use the rectangle $Y^* = (0,l_1)\times(0,G_1)$, the extension by zero and the factors $\Gamma_{[\cdot]_\epsilon}$ to cover R^ϵ and to reflect the oscillations and the variable period. As a consequence we remark that there exist two crucial differences:

1. The support of the functions $\mathcal{T}_\epsilon(\varphi)$ depends on ϵ. (See Figure 2.6). As a matter of fact for a general φ the support of the function $\mathcal{T}_\epsilon(\varphi)$ is given by

$$\begin{aligned} W^\epsilon &= \Big\{(x,y_1,y_2) : x\in(0,1), 0<y_1<l([x]_\epsilon), 0<y_2<G\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \frac{1}{\epsilon}\big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1\big)\Big)\Big\} \\ &= \bigcup_{k=0}^{N_\epsilon} [x_k^\epsilon, x_{k+1}^\epsilon) \times Y_k^\epsilon \subset [0,1]\times Y^*. \end{aligned} \tag{2.2.1}$$

where for $k = 0,1,\dots,N_\epsilon$

$$Y_k^\epsilon = \big\{(y_1,y_2) : 0<y_1<l(x_k^\epsilon), 0<y_2<G\big(x_k^\epsilon + \Gamma_{x_k^\epsilon} y_1, \frac{1}{\epsilon}\big(x_k^\epsilon + \Gamma_{x_k^\epsilon} y_1\big)\big)\big\}.$$

Then, we have to prove that the sequence of these three dimensional domains W^ϵ converges in certain sense to the fixed domain W as ϵ goes to zero. (See Proposition (2.3.4) below).

2. Even if φ is very regular, $\mathcal{T}_\epsilon(\varphi)$ does not inherit regularity as a function of (y_1,y_2). This is a delicate point to obtain the limit of $\mathcal{T}_\epsilon(\frac{\partial\varphi}{\partial x})$ and $\mathcal{T}_\epsilon(\frac{\partial\varphi}{\partial y})$. Indeed, it is not possible to obtain a compactness result using the same arguments as in Theorem 1.2.3. To overcome this difficulty we will use an approach inspired by [2].

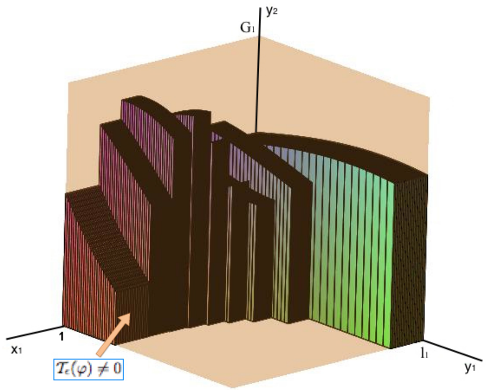


Figure 2.6: The set W^ϵ, the support of $\mathcal{T}_\epsilon(\varphi)$.

In the following proposition we show the main properties of unfolding operator which will be used. Some of them are straightforward and their proofs are omitted.

Proposition 2.2.5. *The unfolding operator $\mathcal{T}_\epsilon$ associated to a general admissible partition $\{x_k^\epsilon\}$ has the following properties:*

i) $\mathcal{T}_\epsilon$ is a linear operator.

ii) $\mathcal{T}_\epsilon(\varphi\psi) = \mathcal{T}_\epsilon(\varphi)\mathcal{T}_\epsilon(\psi)$ and $\mathcal{T}_\epsilon(f \circ \psi) = f \circ \mathcal{T}_\epsilon(\psi)$, $\forall\, \varphi, \psi$ Lebesgue-measurable functions in R^ϵ and $f : \mathbb{R} \to \mathbb{R}$ a continuous function with $f(0) = 0$.

iii) Unfolding criterion for integrals (u.c.i.) :

$$\int_{(0,1)\times Y^*} \frac{1}{l([x]_\epsilon)} \mathcal{T}_\epsilon(\varphi)(x, y_1, y_2)dxdy_1dy_2 = \frac{1}{\epsilon}\int_{R^\epsilon} \varphi(x, y)dxdy, \quad \forall\, \varphi \in L^1(R^\epsilon),$$
$$\frac{1}{\epsilon}\int_{R^\epsilon} l([x]_\epsilon)\varphi(x, y)dxdy = \int_{(0,1)\times Y^*} \mathcal{T}_\epsilon(\varphi)(x, y_1, y_2)dxdy_1dy_2, \quad \forall\, \varphi \in L^1(R^\epsilon). \tag{2.2.2}$$

iv) For every $\varphi \in L^p(R^\epsilon)$, $\mathcal{T}_\epsilon(\varphi) \in L^p\big((0, 1) \times Y^*\big)$, with $1 \le p < \infty$. In addition, the following relationship exists between their norms:

$$\|\mathcal{T}_\epsilon(\varphi)\|_{L^p\left((0,1)\times Y^*\right)} \le \left(\frac{l_1}{\epsilon}\right)^{\frac{1}{p}} \|\varphi\|_{L^p(R^\epsilon)},$$

$$\left(\frac{l_0}{\epsilon}\right)^{\frac{1}{p}} \|\varphi\|_{L^p(R^\epsilon)} \le \|\mathcal{T}_\epsilon(\varphi)\|_{L^p\left((0,1)\times Y^*\right)}.$$

In the special case $p = \infty$,

$$\|\mathcal{T}_\epsilon(\varphi)\|_{L^\infty\left((0,1)\times Y^*\right)} = \|\varphi\|_{L^\infty(R^\epsilon)}.$$

v) Let $\psi \in C^\infty_\#(W)$. We define the sequence $\{\psi^\epsilon\}$ by

$$\psi^\epsilon(x,y) = \psi\big(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}\big) \quad \forall (x,y) \in R^\epsilon,$$

then, $\psi^\epsilon \in C^\infty(\overline{R^\epsilon})$ and for all $(x, y_1, y_2) \in (0,1) \times Y^*$

$$\begin{aligned}&\mathcal{T}_\epsilon(\psi^\epsilon)(x, y_1, y_2)\\ &= \widetilde{\psi}\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \frac{[x]_\epsilon + \Gamma_{[x]_\epsilon} y_1}{\epsilon}, y_2\Big)\chi^\epsilon\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \epsilon y_2\Big)\chi_{\big(0,l([x]_\epsilon)\big)}(y_1).\end{aligned}$$

Proof. *i)* Immediate from the definition of the unfolding operator.

ii) Simple consequence of definition of the unfolding operator.

iii) Let $\varphi \in L^1(R^\epsilon)$. The proof is similar for both equalities so that we will see only the first one:

$$\begin{aligned}&\int_{(0,1)\times Y^*} \frac{1}{l([x]_\epsilon)} \mathcal{T}_\epsilon(\varphi)(x, y_1, y_2)\, dx dy_1 dy_2\\
&= \sum_{k=0}^{N_\epsilon} \int_{(x_k^\epsilon, x_{k+1}^\epsilon)\times Y^*} \frac{1}{l(x_k^\epsilon)} \widetilde{\varphi}\Big(x_k^\epsilon + \Gamma_{x_k^\epsilon} y_1, \epsilon y_2\Big)\chi_{(0,l(x_k^\epsilon))}(y_1)\, dx dy_1 dy_2\\
&= \sum_{k=0}^{N_\epsilon} (x_{k+1}^\epsilon - x_k^\epsilon) \int_{\big(0,l(x_k^\epsilon)\big)\times\big(0,G_1\big)} \frac{1}{l(x_k^\epsilon)} \widetilde{\varphi}\Big(x_k^\epsilon + \Gamma_{x_k^\epsilon} y_1, \epsilon y_2\Big)\, dy_1 dy_2\\
&= \sum_{k=0}^{N_\epsilon} \frac{1}{\epsilon} \int_{(x_k^\epsilon, x_{k+1}^\epsilon)\times(0,\epsilon G_1)} \widetilde{\varphi}(x,y)\, dx dy\\
&= \frac{1}{\epsilon} \int_{(0,1)\times(0,\epsilon G_1)} \widetilde{\varphi}(x,y)\, dy dx = \frac{1}{\epsilon}\int_{R^\epsilon} \varphi(x,y)\, dx dy.\end{aligned}$$

iv) Consequence of iii).

v) First we will see that $\{\psi^\epsilon\}$ is well defined. For every $x \in (0,1)$ there exists $k \in \mathbb{N}$ large enough so that $\frac{x}{\epsilon} - kl(x) \in (0, l(x))$. Furthermore, since $(x,y) \in R^\epsilon$ we have that $0 < y < \epsilon G\big(x, \frac{x}{\epsilon}\big) = \epsilon G\big(x, \frac{x}{\epsilon} - kl(x)\big)$. Hence, $\big(\frac{x}{\epsilon} - kl(x), \frac{y}{\epsilon}\big) \in Y^*(x)$ and $\big(x, \frac{x}{\epsilon} - kl(x), \frac{y}{\epsilon}\big) \in W$. Finally, since for each fixed $x \in (0,1)$ the function ψ is periodic of period $l(x)$ in the variable y_1, $\psi(x, y_1 + l(x), y_2) = \psi(x, y_1, y_2)$, we can conclude that the function is correctly defined

$$\psi^\epsilon(x,y) = \psi\big(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}\big) = \psi\big(x, \frac{x}{\epsilon} - kl(x), \frac{y}{\epsilon}\big).$$

Now, it is obvious that $\psi^\epsilon \in C^\infty(\overline{R^\epsilon})$ from the regularity of ψ and in view of the Definition 2.2.4 we have

$$\mathcal{T}_\epsilon(\psi^\epsilon)(x, y_1, y_2) = \begin{cases} \widetilde{\psi^\epsilon}\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \epsilon y_2\Big) & \text{for} \quad y_1 \in \big(0, l([x]_\epsilon)\big),\\ 0 & \text{for} \quad y_1 \in \big(l([x]_\epsilon), l_1\big),\end{cases}$$

for all $(x, y_1, y_2) \in (0,1) \times Y^*$.

Then, due to the definition of ψ^ϵ and since the characteristic function of W^ϵ, see (2.2.1), is given by

$$T_\epsilon(\chi^\epsilon) = \chi^\epsilon\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \epsilon y_2\Big)\chi_{\big(0, l([x]_\epsilon)\big)}(y_1),$$

we get

$$\begin{aligned}
&\mathcal{T}_\epsilon(\psi^\epsilon)(x, y_1, y_2) \\
&= \widetilde{\psi}\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \frac{[x]_\epsilon + \Gamma_{[x]_\epsilon} y_1}{\epsilon}, y_2\Big)\chi^\epsilon\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \epsilon y_2\Big)\chi_{\big(0, l([x]_\epsilon)\big)}(y_1),
\end{aligned}$$

for all $(x, y_1, y_2) \in (0,1) \times Y^*$.

□

Remark 2.2.6. *Properties iii) and v) will be very important when obtaining the homogenized limit problem. As a matter of fact, property iii) (u.c.i property), shows that the unfolding operator preserves the integral of the functions (up to the multiplicative piecewise constant function $\frac{\epsilon}{l([\cdot]_\epsilon)}$) and property v) shows the relationship between test functions.*

Remark 2.2.7. *Observe that due to the order of the height of the thin domain the factor $\frac{1}{\epsilon}$ appears in the criterion for integrals and in property iv). As a matter of fact, throughout this chapter we will consider the rescaled norms $|||\cdot|||_{L^p(R^\epsilon)} = \epsilon^{-1/p}||\cdot||_{L^p(R^\epsilon)}$ and $|||\cdot|||_{W^{1,p}(R^\epsilon)} = \epsilon^{-1/p}||\cdot||_{W^{1,p}(R^\epsilon)}$, for $1 \le p < \infty$, which are the appropriate norms to analyze the convergence properties in shrinking domains, see Notation Section at the beginning of the thesis.*

Remark 2.2.8. *Notice that the constructed unfolding operator removes the integration defect arisen from the cells which are not completely included in R^ϵ. Recall that the unfolding introduced for the periodic case, see Definition 1.1.3, is not defined in the "incomplete" cells. However, this new extension solves this problem and, as a consequence, it conserves the integrals which will simplify the proofs and removes the necessity of introducing a criterion for passing to the limit as in Definition 1.1.6*

2.3. Convergence properties of the unfolding operator

In this section we analyze the convergence properties of the unfolding operator defined in the previous section, as $\epsilon \to 0$. Moreover, we state and prove the main result for the homogenization. Theorem 2.3.9 is a compactness result which is essential to obtain the homogenized limit problem in the next section.

To have good convergence properties of the unfolding operator we will need to choose an appropriate admissible partition which is related to the variable period $l(x)$ of the boundary of the thin domain and we will denote this special partition as $l(x)$-partition.

To construct the $l(x)$-partition, first of all, notice that from hypothesis **(H)** the function $x \to \frac{x}{l(x)}$ has the set of critical points of measure zero, $\mu(A) = 0$, and the inverse image of every point is at most a finite set, for all $k \in \mathbb{R}$ the points $x \in (0,1)$ such that $x = kl(x)$ is a finite set.

Definition 2.3.1. *For every $\epsilon > 0$ fixed, we define M^ϵ the largest integer such that $M^\epsilon \epsilon \le \max_{x\in[0,1]} \left\{ \frac{x}{l(x)} \right\}$ and consider the points $x \in (0,1)$ such that $\frac{x}{l(x)} = k\epsilon$, for all $k = 1, 2, \dots, M^\epsilon$ (see Figure 2.7). Hence, the $l(x)$-partition is defined by $\{x_i^\epsilon\}$, $i = 0, 1, \dots, N^\epsilon + 1$, such that*

- *$x_0^\epsilon = 0$ and $x_{N^\epsilon+1}^\epsilon = 1$.*
- *Given x_i^ϵ a point of the partition, $i \ne N^\epsilon + 1$, there exists $k \in \{0, \dots, M^\epsilon\}$ such that $\frac{x_i^\epsilon}{l(x_i^\epsilon)} = k\epsilon$.*
- *Two consecutive points of the partition satisfy*

$$\left| \frac{x_{i+1}}{l(x_{i+1})} - \frac{x_i}{l(x_i)} \right| = \epsilon, \ \ or \quad \left| \frac{x_{i+1}}{l(x_{i+1})} - \frac{x_i}{l(x_i)} \right| = 0$$

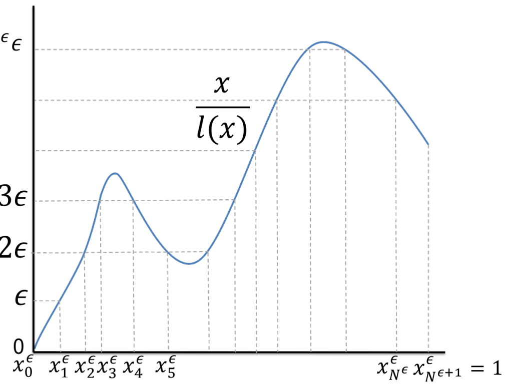


Figure 2.7: The $l(x)$-partition $\{x_k^\epsilon\}$

The $l(x)$-partition is correctly defined and it is an admissible partition by assumption **(H)**. This hypothesis guarantees that inverse image of every $y \in \mathbb{R}$ by the function $\frac{x}{l(x)}$ is at most a finite set and, as a consequence, the $l(x)$-partition has a finite number of points. Moreover, notice that the distance between two consecutive points is not constant but somehow reproduces the locally periodic structure of the thin domain and satisfies that for every $x \in (0,1) \setminus A$ there exist $\epsilon_1 > 0$ and a constant C such that $0 \le x - [x]_\epsilon < C\epsilon$, for $0 < \epsilon < \epsilon_1$.

Remark 2.3.2. *To "justify " the choice of the $l(x)$-partition, consider the particular case where the function which defines the oscillatory boundary of the thin domain is given by*

$$G(x,y) = 2 + \cos\left(\frac{2\pi y}{l(x)}\right)$$

with $l(\cdot)$ verifying the assumption **(H)**. *If we look at the points which are at the top part of the oscillatory boundary, that is, $G(x, x/\epsilon) = 3$, or equivalently,*

$$\cos\Big(\frac{2\pi x}{\epsilon l(x)}\Big) = 1,$$

we obtain a sequence of points which verify

$$\frac{2\pi x}{\epsilon l(x)} = 2\pi k, \ k = 0, 1, 2 \ldots$$

Observe that these points coincide with the points of the $l(x)$-partition. It turns out that these points also work when the amplitude of the oscillations also varies in space since they continue to reproduce in a good way the locally oscillatory behavior of the oscillations.

We start showing the following key property of the $l(x)-$partition $\{x_k^\epsilon\}$ chosen in this section. Recall that the function $G : \mathbb{R} \times \mathbb{R} \to (0, +\infty)$ which defines the oscillatory boundary of the thin domain is a smooth function satisfying that there exist two positive constants G_0, G_1 such that $0 < G_0 \leq G(x, y) \leq G_1$, $\forall\,(x, y) \in (0, 1) \times \mathbb{R}$ and for each $x \in (0, 1)$, the function $G(x, \cdot)$ is $l(x)$-periodic, with $l(\cdot)$ verifying **(H)**.

Proposition 2.3.3. *If $\{x_k^\epsilon\}$ is the $l(x)$-partition and with the notation above, we have the following convergence*

$$G\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \frac{1}{\epsilon}([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1)\Big) - G(x, y_1) \stackrel{\epsilon\to 0}{\longrightarrow} 0 \ \ a.e\ (x, y_1) \in W_0. \tag{2.3.1}$$

Proof. In order to show (2.3.1), we observe that since $G(x, \cdot)$ is $l(x)$-periodic and all the points constructed in Definition 2.3.1 satisfy $\frac{[x]_\epsilon}{\epsilon l([x]_\epsilon)} = k^\epsilon \in \mathbb{N}$ we have

$$\begin{aligned} &G\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \frac{1}{\epsilon}\big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1\big)\Big) \\ &\quad = G\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \frac{1}{\epsilon}\big([x]_\epsilon - l\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1\Big)\frac{[x]_\epsilon}{l([x]_\epsilon)} + \Gamma_{[x]_\epsilon} y_1\big)\Big). \end{aligned}$$

Then, using the regularity properties of $G(\cdot, \cdot)$ and $l(\cdot)$, see **(H)**, we only have to prove the following convergences for almost every $x \in (0, 1)$ and $y_1 \in (0, l(x))$:

$$[x]_\epsilon \stackrel{\epsilon\to 0}{\longrightarrow} x, \ \text{a.e } x \in (0, 1). \tag{2.3.2}$$

$$[x]_\epsilon + \Gamma_{[x]_\epsilon} y_1 \stackrel{\epsilon\to 0}{\longrightarrow} x, \ \text{a.e } (x, y_1) \in W_0. \tag{2.3.3}$$

$$\frac{1}{\epsilon}\Big([x]_\epsilon - l\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1\Big)\frac{[x]_\epsilon}{l([x]_\epsilon)} + \Gamma_{[x]_\epsilon} y_1\Big) \stackrel{\epsilon\to 0}{\longrightarrow} y_1, \ \text{a.e } (x, y_1) \in W_0. \tag{2.3.4}$$

Since the hypothesis **(H)** guarantees that the set A has null measure it is enough to prove the convergences of the Proposition for all $x \in (0,1) \setminus A$.

Let x be a point in $(0,1) \setminus A$, from the definition of the $l(x)-$partition we know that there exist $\epsilon_1 > 0$ and a constant C such that $x \in (x_k^\epsilon, x_{k+1}^\epsilon)$ and $0 < x_{k+1}^\epsilon - x_k^\epsilon < C\epsilon$, for some $k \in \{0, 1, 2, \dots, N^\epsilon + 1\}$ and for all $0 < \epsilon < \epsilon_1$. Therefore, the convergences (2.3.2) and (2.3.3) are immediate from the definition of $[x]_\epsilon$ and $\Gamma_{[x]_\epsilon}$.

In order to prove (2.3.4) we also assume that $x \in (0,1) \setminus A$. From the definition of the $l(x)-$partition we know that for ϵ small enough there exists $p^\epsilon \in \{0, 1, \dots, M^\epsilon\}$ such that $[x]_\epsilon = x_k^\epsilon = \epsilon p^\epsilon l(x_k^\epsilon)$ and $x_{k+1}^\epsilon = \epsilon(p^\epsilon + 1) l(x_{k+1}^\epsilon)$, in case of the function $\frac{x}{l(x)}$ increases in the small interval $(x_k^\epsilon, x_{k+1}^\epsilon)$, or $x_{k+1}^\epsilon = \epsilon(p^\epsilon - 1) l(x_{k+1}^\epsilon)$ in case $\frac{x}{l(x)}$ decreases. We suppose that $x_{k+1}^\epsilon = \epsilon(p^\epsilon + 1) l(x_{k+1}^\epsilon)$, since the proof is analogous for the other case.

We study the limit of the following two terms:

(a). Limit of $\frac{1}{\epsilon}\Gamma_{[x]_\epsilon} y_1$.

Since $x_{k+1}^\epsilon = \epsilon(p^\epsilon + 1) l(x_{k+1}^\epsilon)$ and $x_k^\epsilon = \epsilon p^\epsilon l(x_k^\epsilon)$ we have

$$
\begin{aligned}
\frac{1}{\epsilon}\Gamma_{[x]_\epsilon} y_1 &= \frac{x_{k+1}^\epsilon - x_k^\epsilon}{\epsilon l(x_k^\epsilon)} y_1 = p^\epsilon \frac{y_1}{l(x_k^\epsilon)}\big(l(x_{k+1}^\epsilon) - l(x_k^\epsilon)\big) + \frac{y_1}{l(x_k^\epsilon)} l(x_{k+1}^\epsilon) \\
&= \frac{x_k^\epsilon}{\epsilon l(x_k^\epsilon)^2} y_1 \big(l(x_{k+1}^\epsilon) - l(x_k^\epsilon)\big) + \frac{y_1}{l(x_k^\epsilon)} l(x_{k+1}^\epsilon) \\
&= \frac{x_k^\epsilon}{l(x_k^\epsilon)} y_1 \frac{l(x_{k+1}^\epsilon) - l(x_k^\epsilon)}{x_{k+1}^\epsilon - x_k^\epsilon} \frac{x_{k+1}^\epsilon - x_k^\epsilon}{\epsilon l(x_k^\epsilon)} + \frac{y_1}{l(x_k^\epsilon)} l(x_{k+1}^\epsilon) \\
&= \frac{x_k^\epsilon}{l(x_k^\epsilon)} \frac{l(x_{k+1}^\epsilon) - l(x_k^\epsilon)}{x_{k+1}^\epsilon - x_k^\epsilon} \frac{1}{\epsilon}\Gamma_{[x]_\epsilon} y_1 + \frac{y_1}{l(x_k^\epsilon)} l(x_{k+1}^\epsilon).
\end{aligned}
$$

Consequently we obtain

$$
\Big(1 - \frac{x_k^\epsilon}{l(x_k^\epsilon)} \frac{l(x_{k+1}^\epsilon) - l(x_k^\epsilon)}{x_{k+1}^\epsilon - x_k^\epsilon}\Big) \frac{1}{\epsilon}\Gamma_{[x]_\epsilon} y_1 = \frac{y_1}{l(x_k^\epsilon)} l(x_{k+1}^\epsilon).
$$

Moreover, we can prove that

$$
1 - \frac{x_k^\epsilon}{l(x_k^\epsilon)} \frac{l(x_{k+1}^\epsilon) - l(x_k^\epsilon)}{x_{k+1}^\epsilon - x_k^\epsilon} \neq 0 \quad \text{for } \epsilon \text{ sufficiently small.}
$$

If it were not true, then there exists a sequence $\{\epsilon_i\}_{i\in\mathbb{N}}$ such that $\epsilon_i > \epsilon_{i+1} > 0$ for any $i \in \mathbb{N}$, $\lim_{i\to\infty} \epsilon_i = 0$ and

$$
1 - \frac{x_k^{\epsilon_i}}{l(x_k^{\epsilon_i})} \frac{l(x_{k+1}^{\epsilon_i}) - l(x_k^{\epsilon_i})}{x_{k+1}^{\epsilon_i} - x_k^{\epsilon_i}} = 0, \quad \forall i \in \mathbb{N}.
$$

Hence, we have

$$
\frac{l(x_{k+1}^{\epsilon_i}) - l(x_k^{\epsilon_i})}{x_{k+1}^{\epsilon_i} - x_k^{\epsilon_i}} = \frac{l(x_k^{\epsilon_i})}{x_k^{\epsilon_i}}, \quad \forall i \in \mathbb{N}.
$$

Passing to the limit as $i \to \infty$ we obtain

$$l'(x) = \frac{l(x)}{x},$$

This contradicts the fact that $x \notin A$.

Hence we can write

$$\frac{1}{\epsilon}\Gamma_{[x]_\epsilon} y_1 = \Big(1 - \frac{x_k^\epsilon}{l(x_k^\epsilon)} \frac{l(x_{k+1}^\epsilon) - l(x_k^\epsilon)}{x_{k+1}^\epsilon - x_k^\epsilon}\Big)^{-1} \frac{y_1}{l(x_k^\epsilon)} l(x_{k+1}^\epsilon). \tag{2.3.5}$$

Taking into account the convergence (2.3.2), the hypothesis **(H)** and $x \notin A$ we may pass to the limit at the right-hand side of the equality (2.3.5) to obtain

$$\frac{1}{\epsilon}\Gamma_{[x]_\epsilon} y_1 \overset{\epsilon \to 0}{\longrightarrow} \Big(1 - \frac{x}{l(x)} l'(x)\Big)^{-1} y_1. \tag{2.3.6}$$

(b). Limit of $\frac{1}{\epsilon}\Big([x]_\epsilon - l\big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1\big)\frac{[x]_\epsilon}{l([x]_\epsilon)}\Big)$**.**

First, we write this second term as

$$\frac{1}{\epsilon}\Big([x]_\epsilon - l([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1)\frac{[x]_\epsilon}{l([x]_\epsilon)}\Big) = -\frac{x_k^\epsilon}{l(x_k^\epsilon)} \frac{l(x_k^\epsilon + \Gamma_{x_k^\epsilon} y_1) - l(x_k^\epsilon)}{\Gamma_{x_k^\epsilon} y_1} \frac{\Gamma_{x_k^\epsilon} y_1}{\epsilon}$$

Now it is possible to pass to the limit in the right-hand side by using the convergence (2.3.6). Hence, from convergences (2.3.2), (2.3.3) and (2.3.6) we have

$$\frac{1}{\epsilon}\Big([x]_\epsilon - l([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1)\frac{[x]_\epsilon}{l([x]_\epsilon)}\Big) \overset{\epsilon \to 0}{\longrightarrow} -\frac{x}{l(x)} l'(x)\Big(1 - \frac{x}{l(x)} l'(x)\Big)^{-1} y_1. \tag{2.3.7}$$

Therefore, combining convergences (2.3.6) and (2.3.7), we obtain (2.3.4)

$$\begin{aligned}
&\frac{1}{\epsilon}\big([x]_\epsilon - l([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1)\frac{[x]_\epsilon}{l([x]_\epsilon)} + \Gamma_{[x]_\epsilon} y_1\big) \\
&\quad \overset{\epsilon \to 0}{\longrightarrow} -\frac{x}{l(x)} l'(x)\Big(1 - \frac{x}{l(x)} l'(x)\Big)^{-1} y_1 + \Big(1 - \frac{x}{l(x)} l'(x)\Big)^{-1} y_1 = y_1.
\end{aligned}$$

This concludes the proof of the proposition. □

We show now that the domains W^ϵ, see (2.2.1), converge to the domain W in the sense that the characteristic functions converge strongly in L^p, $1 \le p < \infty$. Recall that we denote by χ the characteristic function of W and χ^ϵ is the characteristic function of R^ϵ. Therefore, $\mathcal{T}_\epsilon(\chi^\epsilon)$ is the characteristic function of W^ϵ. The fact that W^ϵ " approaches " W is expressed in the following result.

Proposition 2.3.4. *With the notations above and if $\mathcal{T}_\epsilon$ is the unfolding operator associated to the $l(x)$-partition $\{x_k^\epsilon\}$, we have*

$$\mathcal{T}_\epsilon(\chi^\epsilon) \longrightarrow \chi \quad s - L^p\big((0,1) \times Y^*\big), \text{ for } 1 \le p < \infty.$$

Proof. Since $T_\epsilon(\chi^\epsilon)$ and χ are uniformly bounded functions (they are actually bounded by the constant 1), it is enough to prove the convergence for $p = 1$. Considering the set represented by each characteristic function we can write:

$$\begin{aligned}
&\|\mathcal{T}_\epsilon(\chi^\epsilon) - \chi\|_{L^1\left((0,1)\times Y^*\right)} = \\
&= \int_0^1 \int_{Y*} |\chi^\epsilon\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \epsilon y_2\Big)\chi_{(0,l([x]_\epsilon))}(y_1) - \chi(x, y_1, y_2)| dy_1 dy_2 dx \\
&= \int_0^1 \int_0^{l([x]_\epsilon)} \int_0^{G\left([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \frac{1}{\epsilon}\left([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1\right)\right)} |1 - \chi(x, y_1, y_2)| dy_2 dy_1 dx \\
&\quad + \int_0^1 \int_0^{l([x]_\epsilon)} \int_{G\left([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \frac{1}{\epsilon}\left([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1\right)\right)}^{G_1} |\chi(x, y_1, y_2)| dy_2 dy_1 dx \\
&\quad + \int_0^1 \int_{l([x]_\epsilon)}^{l_1} \int_0^{G_1} |\chi(x, y_1, y_2)|\, dy_2 dy_1 dx.
\end{aligned}$$

With the convergence

$$l([x]_\epsilon) - l(x) \overset{\epsilon \to 0}{\longrightarrow} 0, \text{ a.e } x \in (0,1), \tag{2.3.8}$$

which follows easily from the convergence (2.3.2) and the continuity of the function $l(\cdot)$, together with (2.3.1) and with the aid of the Lebesgue's Dominated Convergence Theorem, we easily prove that $\|\mathcal{T}_\epsilon(\chi^\epsilon) - \chi\|_{L^1\left((0,1)\times Y^*\right)} \overset{\epsilon \to 0}{\longrightarrow} 0$.

□

Remark 2.3.5. *Note that the choice of the points of the $l(x)$-partition is a key point to obtain the convergence of the domains. Somehow, the fact that the partition suitably reflects the geometry of the oscillating domain is very critical to obtain the convergence result. For example, let's see that things may go wrong if we do not choose wisely the admissible partition, even in the purely periodic case. If we assume that the oscillatory boundary of the thin domain is given by a function $g : \mathbb{R} \to \mathbb{R}$ L-periodic and we consider the partition*

$$x_0^\epsilon = 0 < x_1^\epsilon = \epsilon L_1 < x_2^\epsilon = 2\epsilon L_1 < \cdots < x_{N^\epsilon}^\epsilon = N^\epsilon L_1 < x_{N^\epsilon+1}^\epsilon = 1,$$

where L_1 is rationally independent of L. Notice that this partition is an admissible one, but in this case, Proposition 2.3.3 does not hold, precisely because L_1 and L are rationally independent. Indeed, since $[x]_\epsilon = \epsilon k^\epsilon L_1$ for some integer $k^\epsilon \in \left(0, \frac{1}{L_1 \epsilon}\right)$, k^ϵ tends to infinity as $\epsilon \to 0$, $\Gamma_{[x]_\epsilon} = \epsilon$ and the period of g and L_1 are rationally independent we can not ensure the pointwise convergence of the following family of functions

$$g\Big(\frac{1}{\epsilon}\big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1\big)\Big) = g(k^\epsilon L_1 + y_1).$$

Proposition 2.3.6. *Let $\varphi \in L^p(0,1)$, $1 \le p < \infty$. Then if $\mathcal{T}_\epsilon$ is the unfolding operator associated to the $l(x)$-partition, we have*

$$\mathcal{T}_\epsilon(\varphi) \overset{\epsilon \to 0}{\longrightarrow} \varphi\chi \quad s - L^p\big((0,1) \times Y^*\big).$$

Proof. For any $\varphi \in \mathcal{D}(0,1)$ one gets

$$\|\mathcal{T}_\epsilon(\varphi) - \varphi\chi\|^p_{L^p\left((0,1)\times Y^*\right)} = \int_{(0,1)\times Y^*} |\mathcal{T}_\epsilon(\varphi) - \varphi\chi|^p dx dy_1 dy_2$$
$$\leq G_1 \int_0^1 \int_0^{l_1} \chi_{(0,l([x]_\epsilon))}(y_1)\left|\varphi\big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1\big) - \varphi(x)\right|^p dy_1 dx,$$

So, since φ is uniformly continuous on $[0,1]$ and taking into account convergences (2.3.3) and (2.3.8) we can apply the Lebesgue's Dominated Convergence Theorem to obtain

$$\mathcal{T}_\epsilon(\varphi) \longrightarrow \varphi\chi \quad \mathrm{s} - L^p\big((0,1)\times Y^*\big), \qquad \forall \varphi \in \mathcal{D}(0,1). \tag{2.3.9}$$

By density, if $\varphi \in L^p(0,1)$, let $\varphi_k \in \mathcal{D}(0,1)$ such that $\varphi_k \to \varphi$ in $L^p(0,1)$. Then, we have

$$\|\mathcal{T}_\epsilon(\varphi) - \varphi\chi\|_{L^p\left((0,1)\times Y^*\right)} \leqslant \|\mathcal{T}_\epsilon(\varphi) - \mathcal{T}_\epsilon(\varphi_k)\|_{L^p\left((0,1)\times Y^*\right)}$$
$$+\|\mathcal{T}_\epsilon(\varphi_k) - \varphi_k\chi\|_{L^p\left((0,1)\times Y^*\right)} + \|\varphi_k\chi - \varphi\chi\|_{L^p\left((0,1)\times Y^*\right)}$$

from which the result is straightforward in view of the property iv) in Proposition 2.2.5 and convergence (2.3.9). □

In the following proposition we establish a link between test functions in R^ϵ and test functions in W.

Proposition 2.3.7. *Let $\psi = \psi(x, y_1, y_2)$ be a $C^\infty_\#(W)$ function. We define the following function $\psi^\epsilon \in C^\infty(\overline{R^\epsilon})$ by*

$$\psi^\epsilon(x,y) = \psi\big(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}\big) \quad \forall (x,y) \in R^\epsilon.$$

Then if $\mathcal{T}_\epsilon$ is the unfolding operator associated to the $l(x)$-partition

$$\mathcal{T}_\epsilon(\psi^\epsilon) \overset{\epsilon\to 0}{\longrightarrow} \psi\chi \quad s - L^p\big((0,1)\times Y^*\big), \forall 1 \leq p < \infty.$$

Proof. We have already shown that ψ^ϵ is well defined, see Proposition 2.2.5, v). Then, we only have to prove the convergence.

$$\|\mathcal{T}_\epsilon(\psi^\epsilon) - \psi\chi\|^p_{L^p\left((0,1)\times Y^*\right)} =$$
$$= \int_0^1 \int_{Y*} \left|\psi^\epsilon\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \epsilon y_2\Big)\mathcal{T}_\epsilon(\chi^\epsilon)(x,y_1,y_2) - \psi(x,y_1,y_2)\chi(x,y_1,y_2)\right|^p dy_1 dy_2 dx$$
$$= \int_0^1 \int_{Y*} \left|\psi\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \frac{1}{\epsilon}\big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1\big), y_2\Big)\mathcal{T}_\epsilon(\chi^\epsilon) - \psi\chi\right|^p dy_1 dy_2 dx.$$

Since ψ is continuous on W and taking into account the convergences (2.3.3) and (2.3.4) we have a.e on W

$$\left|\psi\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \frac{1}{\epsilon}\Big([x]_\epsilon - l\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1\Big)\frac{[x]_\epsilon}{l([x]_\epsilon)} + \Gamma_{[x]_\epsilon} y_1\Big), y_2\Big) - \psi(x,y_1,y_2)\right| \overset{\epsilon\to 0}{\longrightarrow} 0. \tag{2.3.10}$$

Hence, due to (2.3.10), Proposition 2.3.4 and by applying the Lebesgue's Dominated Convergence Theorem we obtain the result. □

So far we have only considered functions in $L^p(R^\epsilon)$. Now we study the behavior of sequences in $W^{1,p}(R^\epsilon)$. We will prove a compactness result which allows us to obtain the limit of the unfolded derivatives. Before that, we need to state and prove a technical Lemma which we will use later on.

Lemma 2.3.8. *Assume $1 \le p < \infty$. For any function $\theta(\cdot) \in W_0^{1,p}(0,1)$ there exists a function ψ in $L^p\Big((0,1); W_\#^{1,p}\big(Y^*(x)\big)\Big)$ such that*

$$\psi(x, y_1, y_2) = \psi(x, y_2), \text{ in } W$$

$$\psi(x, y_2) = 0 \quad \text{on} \quad \partial W \setminus B_l,$$

$$\frac{1}{l(x)} \int_{Y^*(x)} \psi(x, y_2) dy_1 dy_2 = \theta(x), \quad \forall x \in (0,1),$$

$$\|\psi\|_{L^p(W)} \leqslant C \|\theta\|_{L^p(0,1)},$$

where B_l is the following lateral boundary of W

$$B_l = \{(x, 0, y_2) : x \in (0,1),\ 0 < y_2 < G(x,0)\} \cup \{(x, l(x), y_2) : x \in (0,1),\ 0 < y_2 < G(x, l(x))\}.$$

Proof. Let us consider the cell $Y_0^* = (0, l_0) \times (0, G_0) \subset Y^*(x),\ \forall x \in (0,1)$. We define the following auxiliary problem:

$$\begin{cases} -\dfrac{\partial^2 v}{\partial {y_2}^2} = 1 & \text{in } Y_0^* \\ \dfrac{\partial v}{\partial y_1} = 0 & \text{in } Y_0^* \\ v = 0 & \text{on } A_1 \cup A_2 \end{cases}$$

where A_1 is the upper boundary and A_2 is the lower boundary of Y_0^*.

This problem admits an unique, nonzero solution that we can obtain explicitly:

$$v(y_2) = -\frac{y_2^2}{2} + \frac{G_0}{2} y_2, \quad \forall y_2 \in (0, G_0).$$

Then we define the function ψ by:

$$\psi(x, y_1, y_2) = \frac{l_0}{\int_{Y_0^*} \left(\frac{\partial v}{\partial y_2}\right)^2 dy_1 dy_2} \widetilde{v}(y_2) \theta(x), \quad \forall (x, y_1, y_2) \in W.$$

It is easy to see that ψ satisfies all the properties of the Lemma:

- It is straightforward from the definition that ψ does not depend on y_1. Note that
$$\frac{\partial \psi}{\partial y_1} = 0.$$

- Since θ has trace zero, recall that $\theta \in W_0^{1,p}(0,1)$, and v is zero on the upper and lower boundary of Y_0^* we have
$$\widetilde{v}(y_2)\theta(x) = 0 \quad \text{on} \quad \partial W \setminus B_l.$$
Consequently, $\psi(x,y_2) = 0$ on $\partial W \setminus B_l$.

- Since $\int_{Y_0^*} \left(\frac{\partial v}{\partial y_2}\right)^2 dy_1 dy_2 = \int_{Y_0^*} v\, dy_1 dy_2$ it follows straightforward that
$$\frac{1}{l(x)} \int_{Y^*(x)} \psi(x,y_2) dy_1 dy_2 = \theta(x), \quad \forall x \in (0,1).$$

- Taking into account that $W \subset (0,1) \times Y^*$ and the definition of v one gets
$$\|\psi\|^p_{L^p(W)} \le \int_0^1 |\theta|^p \, dx \int_{Y^*} \Big| \frac{l_0}{\int_{Y_0^*} \left(\frac{\partial v}{\partial y_2}\right)^2 dy_1 dy_2} \widetilde{v}(y_2) \Big|^p dy_2 \le C \|\theta\|^p_{L^p(0,1)}.$$

Therefore, the proof is complete. □

We are now in position to state the main result of this section.

Theorem 2.3.9. *Let $\varphi^\epsilon \in W^{1,p}(R^\epsilon)$ for $1 < p < \infty$, with the rescaled norm $|||\varphi^\epsilon|||_{W^{1,p}(R^\epsilon)} = \epsilon^{-1/p} \|\varphi^\epsilon\|_{W^{1,p}(R^\epsilon)}$ uniformly bounded. Then, if $\mathcal{T}_\epsilon$ is the unfolding operator associated to the $l(x)$-partition*

i) There exists a function φ in $W^{1,p}(0,1)$ such that, up to subsequences:
$$\mathcal{T}_\epsilon(\varphi^\epsilon) \overset{\epsilon \to 0}{\rightharpoonup} \varphi\chi \quad w - L^p\big((0,1) \times Y^*\big),$$
where χ is the characteristic function of W.

ii) There exists a function φ_1 in $L^p\Big((0,1); W^{1,p}_{\#}(Y^*(x))\Big)$ such that, up to subsequences:

$$\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial x}\Big) \overset{\epsilon \to 0}{\rightharpoonup} \xi_0(x,y_1,y_2) = \begin{cases} \frac{\partial \varphi}{\partial x}(x) + l(x)\frac{\partial \varphi_1}{\partial y_1}(x,y_1,y_2) & \text{for} \quad (x,y_1,y_2) \in W \\ 0 \qquad \text{for} \quad (x,y_1,y_2) \in (0,1) \times Y^* \setminus W. \end{cases}$$

$$\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial y}\Big) \overset{\epsilon \to 0}{\rightharpoonup} \xi_1(x,y_1,y_2) = \begin{cases} l(x)\frac{\partial \varphi_1}{\partial y_2}(x,y_1,y_2) & \text{for} \quad (x,y_1,y_2) \in W \\ 0 \qquad \text{for} \quad (x,y_1,y_2) \in (0,1) \times Y^* \setminus W. \end{cases}$$

Proof. Since $|||\varphi^\epsilon|||_{W^{1,p}(R^\epsilon)}$ is uniformly bounded in ϵ, using Proposition 1.1.4, iv), we deduce that $\|\mathcal{T}_\epsilon(\varphi^\epsilon)\|_{L^p\left((0,1)\times Y^*\right)}$, $\|\mathcal{T}_\epsilon(\frac{\partial \varphi^\epsilon}{\partial x})\|_{L^p\left((0,1)\times Y^*\right)}$ and $\|\mathcal{T}_\epsilon(\frac{\partial \varphi^\epsilon}{\partial y})\|_{L^p\left((0,1)\times Y^*\right)}$ are also uniformly bounded. Therefore, we can extract a subsequence (that we will still index it by ϵ) and obtain functions $\hat{\varphi}, \xi_0, \xi_1 \in L^p\big((0,1) \times Y^*\big)$, such that

$$\begin{aligned}
&\mathcal{T}_\epsilon(\varphi^\epsilon) \overset{\epsilon\to 0}{\rightharpoonup} \hat{\varphi} \quad \text{w} - L^p\big((0,1)\times Y^*\big),\\
&\mathcal{T}_\epsilon(\frac{\partial \varphi^\epsilon}{\partial x}) \overset{\epsilon\to 0}{\rightharpoonup} \xi_0 \quad \text{w} - L^p\big((0,1)\times Y^*\big),\\
&\mathcal{T}_\epsilon(\frac{\partial \varphi^\epsilon}{\partial y}) \overset{\epsilon\to 0}{\rightharpoonup} \xi_1 \quad \text{w} - L^p\big((0,1)\times Y^*\big).
\end{aligned} \tag{2.3.11}$$

i) To check that $\hat{\varphi}$ is zero outside W is obvious from Definition 2.2.4 and Proposition 2.3.4.

Now we prove that $\hat{\varphi}$ does not depend on (y_1, y_2) in W. For this, let $\Psi = (\psi_1, \psi_2)$ be a function in $[\mathcal{D}(W)]^2$. We also denote by Ψ the extension by zero and notice that it belongs to $[\mathcal{D}\big((0,1)\times Y^*\big)]^2$. Using Proposition 2.2.5, v), we can define $\Psi^\epsilon \equiv ({\psi_1}^\epsilon, {\psi_2}^\epsilon) \in [\mathcal{D}(R^\epsilon)]^2$, where

$${\psi_i}^\epsilon(x,y) = \psi_i(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}) \ \forall\, (x,y) \in R^\epsilon, \quad i = 1,2.$$

In addition, set

$${\theta_1}^\epsilon(x,y) = \frac{\partial \psi_1}{\partial x}(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}),\ {\theta_2}^\epsilon(x,y) = \frac{\partial \psi_1}{\partial y_1}(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}) \text{ and } {\theta_3}^\epsilon(x,y) = \frac{\partial \psi_2}{\partial y_2}(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}).$$

Integrating by parts, we have

$$\int_{R^\epsilon} \nabla\varphi^\epsilon(x,y)\cdot\Psi(x, \frac{x}{\epsilon}, \frac{y}{\epsilon})dxdy = -\int_{R^\epsilon} \varphi^\epsilon(x,y)\Big({\theta_1}^\epsilon(x,y) + \frac{1}{\epsilon}{\theta_2}^\epsilon(x,y) + \frac{1}{\epsilon}{\theta_3}^\epsilon(x,y)\Big)dxdy.$$

Then, by the criterion for integrals, Proposition 2.2.5, we obtain

$$\begin{aligned}
&\int_{(0,1)\times Y^*} \frac{\epsilon}{l([x]_\epsilon)} \left\{\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial x}\Big)\mathcal{T}_\epsilon(\psi_1^\epsilon) + \mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial y}\Big)\mathcal{T}_\epsilon(\psi_2^\epsilon)\right\} dxdy_1dy_2\\
&= \ -\int_{(0,1)\times Y^*} \frac{1}{l([x]_\epsilon)} \left\{\epsilon\mathcal{T}_\epsilon(\varphi^\epsilon)\mathcal{T}_\epsilon({\theta_1}^\epsilon) + \mathcal{T}_\epsilon(\varphi^\epsilon)\mathcal{T}_\epsilon({\theta_2}^\epsilon) + \mathcal{T}_\epsilon(\varphi^\epsilon)\mathcal{T}_\epsilon({\theta_3}^\epsilon)\right\} dxdy_1dy_2.
\end{aligned}$$

Passing to the limit in both terms with the help of Proposition 2.3.4 and Proposition 2.3.7 we get

$$0 = -\int_W \frac{1}{l(x)}\hat{\varphi}(x, y_1, y_2)\mathrm{div}_{y_1y_2}\Psi(x, y_1, y_2)dxdy_1dy_2, \quad \forall \Psi \in [\mathcal{D}(W)]^2.$$

This implies that $\hat{\varphi}$ does not depend on (y_1, y_2) in W. Then we can conclude that there exists a function $\varphi \in L^p(0,1)$ such that:

$$\hat{\varphi}(x, y_1, y_2) = \varphi(x)\chi(x, y_1, y_2), \quad \forall (x, y_1, y_2) \in (0,1)\times Y^*.$$

We see now that $\varphi \in W^{1,p}(0,1)$. For this, for any function $\theta(x) \in \mathcal{D}(0,1)$ let ψ be the function in $W^{1,q}(W)$, $\frac{1}{p} + \frac{1}{q} = 1$, given by Lemma 2.3.8, that is

$$\frac{\partial \psi}{\partial y_1} = 0 \quad \text{in } W$$

$$\psi(x, y_2) = 0 \quad \text{on} \quad \partial W \setminus B_l,$$
$$\frac{1}{l(x)} \int_{Y^*(x)} \psi(x, y_2) dy_1 dy_2 = \theta(x) \quad \forall x \in (0,1),$$
$$\|\psi\|_{L^q(W)} \leqslant C \|\theta\|_{L^q(0,1)},$$

where B_l is the lateral boundary of W defined in Lemma 2.3.8. Note that the extension by zero of ψ in the direction y_2 belongs to the space $W^{1,q}\big((0,1) \times Y^*\big)$.

We define the sequence $\{\psi^\epsilon\}$ by

$$\psi^\epsilon(x, y) = \psi\Big(x, \frac{y}{\epsilon}\Big), \quad \forall (x, y) \in R^\epsilon.$$

Integrating by parts, we obtain

$$\int_{R^\epsilon} \frac{\partial \varphi^\epsilon}{\partial x}(x, y) \psi^\epsilon(x, y) dx dy = - \int_{R^\epsilon} \varphi^\epsilon(x, y) \frac{\partial \psi^\epsilon}{\partial x}(x, y) dx dy.$$

Then, by the unfolding criterion for integrals, we have

$$\int_{(0,1)\times Y^*} \frac{1}{l([x]_\epsilon)} \mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial x}\Big) \mathcal{T}_\epsilon(\psi^\epsilon) dx dy_1 dy_2 = - \int_{(0,1)\times Y^*} \frac{1}{l([x]_\epsilon)} \mathcal{T}_\epsilon(\varphi^\epsilon) \mathcal{T}_\epsilon\Big(\frac{\partial \psi^\epsilon}{\partial x}\Big) dx dy_1 dy_2. \tag{2.3.12}$$

We can pass to the limit in (2.3.12) using Proposition 2.3.4 , Proposition 2.3.7 and assuming that $\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial x}\Big) \rightharpoonup \xi_0(x, y_1, y_2)$ w-$L^p\big((0,1)\times Y^*\big)$, it will be seen in the second part of Theorem. Thus, we get

$$\int_W \frac{1}{l(x)} \xi_0(x, y_1, y_2) \psi(x, y_2) dx dy_1 dy_2 = - \int_W \frac{1}{l(x)} \varphi(x) \frac{\partial \psi}{\partial x}(x, y_2) dx dy_1 dy_2. \tag{2.3.13}$$

By using Lemma 2.3.8 above, the right-hand side of the equality (2.3.13) becomes $-\int_{(0,1)} \varphi(x) \frac{\partial \theta}{\partial x}(x) dx$, while the left-hand side is a linear continuous form in $\theta(x) \in \mathcal{D}(0,1)$. Indeed, since $\xi_0 \in L^p\big((0,1)\times Y^*\big)$ and $\|\psi\|_{L^q(W)} \leqslant C\|\theta\|_{L^q(0,1)}$ the left-hand side satisfies

$$\Big| \int_W \frac{1}{l(x)} \xi_0(x, y_1, y_2) \psi(x, y_2) dx dy_1 dy_2 \Big| \le C_1 \|\xi_0\|_{L^p(W)} \|\psi\|_{L^q(W)} \le C_2 \|\theta\|_{L^q(0,1)}.$$

This implies that

$$\Big| \int_{(0,1)} \varphi(x) \frac{\partial \theta}{\partial x}(x) dx \Big| \le C_2 \|\theta\|_{L^q(0,1)}, \quad \forall \theta \in \mathcal{D}(0,1).$$

Therefore, we get $\varphi \in W^{1,p}(0,1)$, see [30].

ii) First note that the functions ξ_0, ξ_1 defined by (2.3.11) satisfy $\xi_0 = \xi_1 = 0$ in $(0,1)\times Y^* \backslash W$ follows from the definition of unfolding operator and Proposition 2.3.4.

In order to find the precise form of ξ_0 and ξ_1 in W we argue as follows. Let $\Psi \equiv (\psi_1, \psi_2) \in \left[C^\infty_\#(W)\right]^2$ be a function satisfying

$$\mathrm{div}_{y_1 y_2}\Psi = 0$$

$$\Psi(x, y_1, y_2) \cdot N(x) = 0 \text{ on } B_1(x) \cup B_2(x),$$

$$\psi_1(0, \cdot, \cdot) = \psi_1(1, \cdot, \cdot) = 0,$$

where again $N(x) = (N_1, N_2)$, $B_1(x)$ and $B_2(x)$ are defined in Section 2.1.

From Proposition 2.3.7 we can define $\Psi^\epsilon \equiv ({\psi_1}^\epsilon, {\psi_2}^\epsilon) \in [W^{1,p}(R^\epsilon)]^2$, where

$${\psi_i}^\epsilon(x, y) = \psi_i(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}) \;\; \forall\, (x, y) \in R^\epsilon, \quad i = 1, 2.$$

Then, considering φ as a function in R^ϵ, so that $\nabla\varphi(x, y) = (\varphi'(x), 0)$, and integrating by parts, we have

$$\begin{aligned}
&\int_{R^\epsilon} \Big[\nabla\varphi^\epsilon(x, y) - \nabla\varphi(x)\Big] \cdot \Psi^\epsilon(x, y)\, dxdy - \int_{\partial_{sup} R^\epsilon} \nu^\epsilon \cdot \Psi^\epsilon(\varphi^\epsilon - \varphi)\, dS = \\
&= -\int_{R^\epsilon} \Big[\varphi^\epsilon(x, y) - \varphi(x)\Big] \mathrm{div}_{(x,y)} \Psi^\epsilon(x, y)\, dxdy \\
&= -\int_{R^\epsilon} \Big[\varphi^\epsilon(x, y) - \varphi(x)\Big] \frac{\partial \psi_1}{\partial x}(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}) dxdy \\
&\quad - \int_{R^\epsilon} \frac{1}{\epsilon} \Big[\varphi^\epsilon(x, y) - \varphi(x)\Big] \mathrm{div}_{(y_1,y_2)} \Psi(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}) dxdy \\
&= -\int_{R^\epsilon} \Big[\varphi^\epsilon(x, y) - \varphi(x)\Big] \frac{\partial \psi_1}{\partial x}(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}) dxdy,
\end{aligned} \tag{2.3.14}$$

where $\partial_{sup} R^\epsilon$ is the upper boundary of R^ϵ and ν^ϵ is the unit outward normal to $\partial_{sup} R^\epsilon$ which is given by

$$\nu^\epsilon = \Big(\frac{-\epsilon G_x - G_y}{\sqrt{(\epsilon G_x + G_y)^2 + 1}}, \frac{1}{\sqrt{(\epsilon G_x + G_y)^2 + 1}}\Big),$$

where $G_x = \dfrac{\partial G}{\partial x}$ and $G_y = \dfrac{\partial G}{\partial y}$.

Applying the unfolding criterion for integrals we can write (2.3.14) as:

$$\begin{aligned}
&\int_{(0,1)\times Y^*} \frac{1}{l([x]_\epsilon)} \left\{\Big[\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial x}\Big) - \mathcal{T}_\epsilon\Big(\frac{\partial \varphi}{\partial x}\Big)\Big] \mathcal{T}_\epsilon(\psi_1) + \mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial y}\Big) \mathcal{T}_\epsilon(\psi_2)\right\} dx dy_1 dy_2 \\
&= -\int_{(0,1)\times Y^*} \frac{1}{l([x]_\epsilon)} \Big[\mathcal{T}_\epsilon(\varphi^\epsilon) - \mathcal{T}_\epsilon(\varphi)\Big] \mathcal{T}_\epsilon\Big(\frac{\partial \psi_1}{\partial x}\Big) dx dy_1 dy_2 \\
&\quad + \frac{1}{\epsilon} \int_{\partial_{sup} R^\epsilon} \nu^\epsilon \cdot \Psi^\epsilon(\varphi^\epsilon - \varphi)\, dS.
\end{aligned} \tag{2.3.15}$$

Now, we prove that the boundary term in (2.3.15) vanishes as ϵ tends to zero.

Then, taking into account $\Psi(x, y_1, y_2) \cdot N(x) = 0$ on $B_1(x)$ we have

$$\frac{1}{\epsilon} \int_{\partial_{sup} R^\epsilon} \nu^\epsilon \cdot \Psi^\epsilon (\varphi^\epsilon - \varphi)\, dS = - \int_{\partial_{sup} R^\epsilon} \frac{G_x}{\sqrt{(\epsilon G_x + G_y)^2 + 1}} \psi_1^\epsilon (\varphi^\epsilon - \varphi)\, dS$$

Consequently, by the regularity of the function G and taking into account that $\psi_1 \in C^\infty_\#(W)$ and the definition of ψ_1^ϵ we obtain

$$\begin{aligned} \Big| \frac{1}{\epsilon} \int_{\partial_{sup} R^\epsilon} \nu^\epsilon \cdot \Psi^\epsilon (\varphi^\epsilon - \varphi)\, dS \Big| &\leq C ||\psi_1||_{L^\infty(W)} \int_{\partial_{sup} R^\epsilon} \big|\varphi^\epsilon - \varphi\big|\, dS \\ &\leq C_1 \int_{\partial_{sup} R^\epsilon} \big|\varphi^\epsilon - \varphi\big|\, dS, \end{aligned} \tag{2.3.16}$$

where C and C_1 are independent of ϵ.

Now, we see that $\int_{\partial_{sup} R^\epsilon} \big|\varphi^\epsilon - \varphi\big|\, dS$ tends to zero.

First of all, notice that since $\partial_{sup} R^\epsilon = \{(x, \epsilon G(x, x/\epsilon) | x \in (0,1)\}$ and by the definition of the line integral along a smooth curve we have

$$\int_{\partial_{sup} R^\epsilon} \big|\varphi^\epsilon - \varphi\big|\, dS = \int_0^1 \big|\varphi^\epsilon\big(x, \epsilon G(x, x/\epsilon)\big) - \varphi(x)\big| \left(1 + (\epsilon G_x + G_y)^2\right)^{1/2} dx.$$

Hence, taking into account the regularity of the function G we obtain

$$\int_{\partial_{sup} R^\epsilon} \big|\varphi^\epsilon - \varphi\big|\, dS \leq C \int_0^1 \big|\varphi^\epsilon\big(x, \epsilon G(x, x/\epsilon)\big) - \varphi(x)\big|\, dx,$$

where C is a constant independent of ϵ. In fact, it follows that

$$\begin{aligned} \int_{\partial_{sup} R^\epsilon} \big|\varphi^\epsilon - \varphi\big|\, dS &\leq C \int_0^1 \big|\varphi^\epsilon\big(x, \epsilon G(x, x/\epsilon)\big) - \varphi^\epsilon\big(x, 0\big)\big|\, dx \\ &+ C \int_0^1 \big|\varphi^\epsilon(x, 0) - \varphi(x)\big|\, dx. \end{aligned} \tag{2.3.17}$$

On the one hand, since $\epsilon^{-1/p} \|\varphi^\epsilon\|_{W^{1,p}(R^\epsilon)}$ uniformly bounded in ϵ we get the convergence of the first term

$$\begin{aligned} \int_0^1 \big|\varphi^\epsilon\big(x, \epsilon G(x, x/\epsilon)\big) - \varphi^\epsilon\big(x, 0\big)\big|\, dx &= \int_0^1 \Big| \int_0^{\epsilon G(x, x/\epsilon)} \frac{\partial \varphi^\epsilon}{\partial y}\, dy \Big|\, dx \\ &\leq \int_{R^\epsilon} \Big| \frac{\partial \varphi^\epsilon}{\partial y} \Big| dx dy \xrightarrow{\epsilon \to 0} 0. \end{aligned} \tag{2.3.18}$$

On the other hand, to obtain the convergence of the second term in (2.3.17) we define the function $\hat{\varphi}^\epsilon$ as follows

$$\hat{\varphi}^\epsilon = \varphi^\epsilon(x, \epsilon y), \quad \forall (x, y) \in R_0, \tag{2.3.19}$$

where R_0 is the fixed domain given by $R_0 = \{(x, y) \in \mathbb{R}^2 \,|\, x \in (0,1), 0 < y < G_0\}$.

Since $\hat{\varphi}^\epsilon$ is defined from φ^ϵ performing the change of variables $(x, y) \longrightarrow (x, y/\epsilon)$ and by hypothesis $\epsilon^{-1/p}\|\varphi^\epsilon\|_{W^{1,p}(R^\epsilon)}$ is uniformly bounded it is not difficult to prove that

$$||\hat{\varphi}^\epsilon||_{L^p(R_0)}, \Big|\Big|\frac{\partial \hat{\varphi}^\epsilon}{\partial x}\Big|\Big|_{L^p(R_0)} \text{ and } \frac{1}{\epsilon}\Big|\Big|\frac{\partial \hat{\varphi}^\epsilon}{\partial y}\Big|\Big|_{L^p(R_0)} \leq C, \quad \forall \epsilon > 0.$$

Then, we can extract a subsequence of $\{\hat{\varphi}^\epsilon\} \subset W^{1,p}(R_0)$, denoted again by $\hat{\varphi}^\epsilon$ and obtain a function $\varphi_0 \in W^{1,p}(R_0)$, such that

$$\hat{\varphi}^\epsilon \rightharpoonup \varphi_0 \quad w - W^{1,p}(R_0), \text{ and } \frac{\partial \hat{\varphi}^\epsilon}{\partial y} \to 0 \quad s - L^p(R_0), \tag{2.3.20}$$

as $\epsilon \to 0$ for some $\varphi_0 \in W^{1,p}(R_0)$. As a consequence of these limits we have that φ_0 does not depend on the variable y, $\varphi_0 \in W^{1,p}(0, 1)$, and the restriction of $\hat{\varphi}^\epsilon$ to the coordinates axis x converges to φ_0, that is,

$$\lim_{\epsilon \to 0} \|\hat{\varphi}^\epsilon - \varphi_0\|_{L^1(\Gamma)} = 0, \tag{2.3.21}$$

where $\Gamma = \{(x, 0) \in \mathbb{R}^2 \,|\, x \in (0, 1)\}$.

Moreover, we get that $\varphi_0 = \varphi$. Indeed, let $\phi \in \mathcal{D}(0, 1)$, then performing the change of variables $(x, y) \longrightarrow (x, \epsilon y)$, by the definition of the unfolding operator and by the criterion for integrals, see Proposition 1.1.4, we have

$$\begin{aligned}\int_{R_0} \hat{\varphi}^\epsilon \phi \, dxdy &= \frac{1}{\epsilon} \int_0^1 \int_0^{\epsilon G_0} \varphi^\epsilon \phi \, dxdy = \frac{1}{\epsilon} \int_{R^\epsilon} \varphi^\epsilon \phi \chi_{(0, \epsilon G_0)}(y) \, dxdy \\ &= \int_{(0,1) \times Y^*} \frac{1}{l([x]_\epsilon)} \mathcal{T}_\epsilon(\varphi^\epsilon) \mathcal{T}_\epsilon(\phi) \chi_{(0, G_0)}(y_2) dxdy_1dy_2.\end{aligned}$$

Then, due to convergences (2.3.20), convergence i) proved in the first part of this theorem and Proposition 1.1.10 we can pass to the limit on the left hand-side and the right hand-side to obtain

$$\int_{R_0} \varphi_0 \, \phi \, dxdy = \int_W \frac{1}{l(x)} \varphi \, \phi \, \chi_{(0, G_0)}(y_2) \, dxdy_1dy_2, \quad \forall \phi \in \mathcal{D}(0, 1).$$

Hence, since φ_0, ϕ and φ depend only on the variable x it follows that

$$\int_0^1 \varphi_0 \, \phi \, dx = \int_0^1 \varphi \, \phi, \quad \forall \phi \in \mathcal{D}(0, 1),$$

which implies that $\varphi_0 = \varphi$.

Thus, taking into account that $\varphi_0 = \varphi$ and from definition (2.3.19) and convergence (2.3.21) we obtain the convergence of the second summand of (2.3.17)

$$\int_0^1 \big|\varphi^\epsilon(x, 0) - \varphi(x)\big| \, dx = \int_0^1 \big|\hat{\varphi}^\epsilon(x, 0) - \varphi(x)\big| \, dx \to 0, \text{ as } \epsilon \to 0. \tag{2.3.22}$$

Therefore, from (2.3.17), (2.3.18) and (2.3.22) we have proved that

$$\int_{\partial_{sup} R^\epsilon} \big|\varphi^\epsilon - \varphi\big| \overset{\epsilon \to 0}{\longrightarrow} 0. \tag{2.3.23}$$

Finally, in light of (2.3.16) and (2.3.23) we can conclude the boundary term in (2.3.15) tends to zero

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\partial_{sup} R^\epsilon} \nu^\epsilon \Psi^\epsilon (\varphi^\epsilon - \varphi)\, dS = 0. \tag{2.3.24}$$

Passing to the limit in (2.3.15) with the help of Proposition 2.3.4, Proposition 2.3.7, the convergence in i) and (2.3.24) we get

$$\int_W \frac{1}{l(x)} \left\{ \left[\xi_0(x, y_1, y_2) - \frac{\partial \varphi}{\partial x}(x) \right] \psi_1(x, y_1, y_2) + \xi_1(x, y_1, y_2) \psi_2(x, y_1, y_2) \right\} dx dy_1 dy_2 = 0.$$

Hence, we obtain

$$\int_W \left(\frac{1}{l(x)} \left[\xi_0(x, y_1, y_2) - \frac{\partial \varphi}{\partial x}(x) \right], \frac{1}{l(x)} \xi_1(x, y_1, y_2) \right) \cdot (\psi_1, \psi_2)(x, y_1, y_2) dx dy_1 dy_2 = 0. \tag{2.3.25}$$

The Helmholtz decomposition, see [61], yields that the orthogonal of divergence-free functions is exactly the gradients. Then, we can conclude that there exists a function $\varphi_1 \in L^p\Big((0,1); W^{1,p}_{\#}\big(Y^*(x)\big)\Big)$ such that

$$\frac{1}{l(x)} \left(\xi_0(x, y_1, y_2) - \frac{\partial \varphi}{\partial x}(x) \right) = \frac{\partial \varphi_1}{\partial y_1}(x, y_1, y_2) \quad \forall (x, y_1, y_2) \in W,$$

$$\frac{1}{l(x)} \xi_1(x, y_1, y_2) = \frac{\partial \varphi_1}{\partial y_2}(x, y_1, y_2) \quad \forall (x, y_1, y_2) \in W,$$

which ends the proof of the Theorem.

□

Remark 2.3.10. *As we wrote in the introduction, this Theorem is the main tool to obtain the homogenized limit problem. On one hand, it shows that the limit φ lies in $W^{1,p}(0,1)$, which was not clear at all in view of the a priori estimates because $\mathcal{T}_\epsilon(\varphi^\epsilon)$ is defined on a varying set. On the other hand, it allows us to relate the limit of the unfolded derivatives $\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial x}\Big)$ and $\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial y}\Big)$ with the weak derivative of φ. Note that the variable period plays a decisive role in the limit and, as we will see in the next section, it enters into the limit equation.*

2.4. Homogenization of the Neumann problem

In this section we return to the problem (2.0.1) presented in the Introduction and we show how the unfolding operator method adapted to this new situation allows to obtain the homogenized limit problem. We will need the results from the previous sections and in particular the convergence result from Theorem 2.3.9. Therefore, throughout this section we will assume that the unfolding operator $\mathcal{T}_\epsilon$ is the one associated to the $l(x)$-partition.

The variational formulation of (2.0.1) is

$$\begin{cases} \text{Find } u^\epsilon \in H^1(R^\epsilon) \text{ such that} \\ \displaystyle\int_{R^\epsilon} \Big\{ \nabla u^\epsilon \cdot \nabla \varphi + u^\epsilon \varphi \Big\} dxdy = \int_{R^\epsilon} f^\epsilon \varphi dxdy, \\ \forall \varphi \in H^1(R^\epsilon). \end{cases} \tag{2.4.1}$$

From Lax-Milgram Theorem, we have that problem (2.0.1) has a unique solution for each $\epsilon > 0$. We are interested here in analyzing the behavior of the solutions as $\epsilon \to 0$.

Now we are in condition to state and prove the homogenization result.

Theorem 2.4.1. *Let u^ϵ be the solution of problem (2.0.1). Assume that $f^\epsilon \in L^2(R^\epsilon)$ satisfies $|||f^\epsilon|||_{L^2(R^\epsilon)} \leq C$ with C independent of the parameter ϵ and, therefore, there exists $\hat{f} \in L^2(W)$ such that, via subsequences, $\mathcal{T}_\epsilon(f^\epsilon) \rightharpoonup \hat{f}\chi$ weakly in $L^2\big((0,1) \times Y^*\big)$. Then, there exist $u \in H^1(0,1)$ and $u_1 \in L^2\Big((0,1); H^1_\#\big(Y^*(x)\big)\Big)$ such that*

$$\mathcal{T}_\epsilon(u^\epsilon) \overset{\epsilon \to 0}{\rightharpoonup} \hat{u} = u\chi \quad \text{weakly in } L^2\big((0,1) \times Y^*\big), \tag{2.4.2}$$

$$\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big) \overset{\epsilon \to 0}{\rightharpoonup} \xi_0(x, y_1, y_2) = \frac{\partial u}{\partial x}(x) + l(x)\frac{\partial u_1}{\partial y_1}(x, y_1, y_2) \quad \text{weakly in } L^2(W), \tag{2.4.3}$$

$$\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial y}\Big) \overset{\epsilon \to 0}{\rightharpoonup} \xi_1(x, y_1, y_2) = l(x)\frac{\partial u_1}{\partial y_2}(x, y_1, y_2) \qquad \text{weakly in } L^2(W), \tag{2.4.4}$$

and the pair (u, u_1) is the unique solution in $H^1(0,1) \times L^2\Big((0,1); H^1_\#\big(Y^*(x)\big)/\mathbb{R}\Big)$ of the problem

$$\begin{cases} \forall \phi \in H^1(0,1), \forall \psi \in L^2\Big((0,1); H^1_\#\big(Y^*(x)\big)\Big) \\ \displaystyle\int_W \bigg\{ \Big(\frac{1}{l(x)} \frac{\partial u}{\partial x}(x) + \frac{\partial u_1}{\partial y_1}(x, y_1, y_2) \Big) \Big(\frac{\partial \phi}{\partial x}(x) + \frac{\partial \psi}{\partial y_1}(x, y_1, y_2) \Big) \bigg\} dxdy_1dy_2 \\ \quad + \displaystyle\int_W \bigg\{ \frac{\partial u_1}{\partial y_2}(x, y_1, y_2) \frac{\partial \psi}{\partial y_2}(x, y_1, y_2) + \frac{u(x)\phi(x)}{l(x)} \bigg\} dxdy_1dy_2 \\ = \displaystyle\int_W \frac{\hat{f}(x, y_1, y_2)\phi(x)}{l(x)} dxdy_1dy_2. \end{cases} \tag{2.4.5}$$

Equivalently, $u \in H^1(0,1)$ is the unique weak solution of the following Neumann problem, it was obtained through the relation $u_1(x, y_1, y_2) = -X(x)(y_1, y_2)\frac{1}{l(x)}\frac{\partial u}{\partial x}(x)$,

$$\begin{cases} -\big(r(x)u_x\big)_x + \dfrac{|Y^*(x)|}{l(x)} u = f_0, \quad x \in (0,1) \\ u'(0) = u'(1) = 0 \end{cases} \tag{2.4.6}$$

where

$$f_0 = \frac{1}{l(x)} \int_{Y^*(x)} \hat{f} dy_1 dy_2, \tag{2.4.7}$$

$$r(x) = \frac{1}{l(x)} \int_{Y^*(x)} \Big\{1 - \frac{\partial X(x)}{\partial y_1}(y_1, y_2)\Big\} dy_1 dy_2 \tag{2.4.8}$$

and $X(x,\cdot,\cdot)$ *is the unique solution which is* $l(x)$*-periodic in the first variable, of the problem*

$$\begin{cases} -\Delta X(x, y_1, y_2) = 0 \ \textit{in} \ Y^*(x) \\ \dfrac{\partial X(x,\cdot,\cdot)}{\partial N} = 0 \ \textit{on} \ B_2(x) \\ \dfrac{\partial X(x,\cdot,\cdot)}{\partial N} = N_1(x) \ \textit{on} \ B_1(x) \\ \displaystyle\int_{Y^*(x)} X(x, y_1, y_2)\, dy_1 dy_2 = 0 \end{cases} \tag{2.4.9}$$

where $B_1(x)$*,* $B_2(x)$ *and* $N(x) = (N_1(x), N_2(x))$ *are defined in* (2.0.9)*,* (2.0.10)*.*

Proof. We start by establishing a priori estimates of u^ϵ. In fact, taking $\varphi = u^\epsilon$ in the variational formulation (2.4.1), we obtain

$$\Big\|\frac{\partial u^\epsilon}{\partial x}\Big\|^2_{L^2(R^\epsilon)} + \Big\|\frac{\partial u^\epsilon}{\partial y}\Big\|^2_{L^2(R^\epsilon)} + \|u^\epsilon\|^2_{L^2(R^\epsilon)} \leq \|f^\epsilon\|_{L^2(R^\epsilon)} \|u^\epsilon\|_{L^2(R^\epsilon)}. \tag{2.4.10}$$

Consequently,

$$\Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial x}\Big|\Big|\Big|^2_{L^2(R^\epsilon)} + \Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial y}\Big|\Big|\Big|^2_{L^2(R^\epsilon)} + |||u^\epsilon|||^2_{L^2(R^\epsilon)} \leq |||f^\epsilon|||_{L^2(R^\epsilon)} |||u^\epsilon|||_{L^2(R^\epsilon)}. \tag{2.4.11}$$

Taking into account that there exists a constant $C > 0$ such that $|||f^\epsilon|||_{L^2(R^\epsilon)} \leqslant C$, we obtain

$$|||u^\epsilon|||_{L^2(R^\epsilon)}, \Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial x}\Big|\Big|\Big|_{L^2(R^\epsilon)} \text{ and } \Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial y}\Big|\Big|\Big|_{L^2(R^\epsilon)} \leq C \quad \forall \epsilon > 0. \tag{2.4.12}$$

Therefore, the compactness Theorem 2.3.9 implies that there exist $u \in H^1(0,1)$ and $u_1 \in L^2\Big((0,1); H^1_\#\big(Y^*(x)\big)\Big)$ such that

$$\begin{aligned} &\mathcal{T}_\epsilon(u^\epsilon) \overset{\epsilon \to 0}{\rightharpoonup} \hat{u} = u\chi \quad \text{weakly in } \ L^2\big((0,1) \times Y^*\big), \\ &\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big) \overset{\epsilon \to 0}{\rightharpoonup} \xi_0(x, y_1, y_2) = \frac{\partial u}{\partial x}(x) + l(x)\frac{\partial u_1}{\partial y_1}(x, y_1, y_2) \quad \text{weakly in } \ L^2(W), \\ &\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial y}\Big) \overset{\epsilon \to 0}{\rightharpoonup} \xi_1(x, y_1, y_2) = l(x)\frac{\partial u_1}{\partial y_2}(x, y_1, y_2) \qquad \text{weakly in } \ L^2(W). \end{aligned} \tag{2.4.13}$$

We are now in the position of finding the homogenized equations satisfied by u and u_1. Let us apply the unfolding operator to the original variational formulation (2.4.1). For $\phi \in H^1(0,1)$, by the unfolding criterion for integrals (2.2.2), we have

$$\int_{(0,1)\times Y^*} \frac{1}{l([x]_\epsilon)} \Big\{ \mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big)\mathcal{T}_\epsilon\Big(\frac{\partial \phi}{\partial x}\Big) + \mathcal{T}_\epsilon(u^\epsilon)\mathcal{T}_\epsilon(\phi)\Big\} dx dy_1 dy_2 \\ = \int_{(0,1)\times Y^*} \frac{1}{l([x]_\epsilon)} \mathcal{T}_\epsilon(f^\epsilon)\mathcal{T}_\epsilon(\phi) dx dy_1 dy_2.$$

Observe that in this last equality we have taken $\phi \in H^1(0,1)$ and the term including partial derivative with respect to y does not appear. By the convergences of (2.4.13) together with Proposition 2.3.6 we can pass to the limit in the last equality and we obtain the first equation:

$$\int_W \Big\{ \Big(\frac{1}{l(x)}\frac{\partial u}{\partial x}(x) + \frac{\partial u_1}{\partial y_1}(x,y_1,y_2)\Big)\frac{\partial \phi}{\partial x}(x) + \frac{u(x)\phi(x)}{l(x)}\Big\} dx dy_1 dy_2 = \\ = \int_W \frac{\hat{f}(x)\phi(x)}{l(x)} dx dy_1 dy_2, \quad \forall \phi \in H^1(0,1). \quad (2.4.14)$$

We take now as a test function in (2.4.1) the function v^ϵ defined by:

$$v^\epsilon(x,y) = \epsilon \psi\Big(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}\Big), \quad \forall (x,y) \in R^\epsilon,$$

where $\psi \in C^\infty_\#(W)$.

It is obvious from the definition that $v^\epsilon \in H^1(R^\epsilon)$. Furthermore, it satisfies

$$\frac{\partial v^\epsilon}{\partial x} = \epsilon \frac{\partial \psi}{\partial x}\Big(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}\Big) + \frac{\partial \psi}{\partial y_1}\Big(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}\Big),$$

$$\frac{\partial v^\epsilon}{\partial y} = \frac{\partial \psi}{\partial y_2}\Big(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}\Big).$$

Hence, using the properties of the unfolding operator we have

$$\begin{aligned} &\mathcal{T}_\epsilon(v^\epsilon) \xrightarrow{\epsilon\to 0} 0 \quad \text{s-}L^2((0,1)\times Y^*), \\ &\mathcal{T}_\epsilon\Big(\frac{\partial v^\epsilon}{\partial x}\Big) \xrightarrow{\epsilon\to 0} \frac{\partial \psi}{\partial y_1}\chi \quad \text{s-}L^2((0,1)\times Y^*), \\ &\mathcal{T}_\epsilon\Big(\frac{\partial v^\epsilon}{\partial y}\Big) \xrightarrow{\epsilon\to 0} \frac{\partial \psi}{\partial y_2}\chi \quad \text{s-}L^2((0,1)\times Y^*). \end{aligned} \quad (2.4.15)$$

Due to the unfolding criterion for integrals, from the variational formulation (2.4.1) we obtain

$$\int_{(0,1)\times Y^*} \frac{1}{l([x]_\epsilon)} \Big\{ \mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big)\mathcal{T}_\epsilon\Big(\frac{\partial v^\epsilon}{\partial x}\Big) + \mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial y}\Big)\mathcal{T}_\epsilon\Big(\frac{\partial v^\epsilon}{\partial y}\Big) + \mathcal{T}_\epsilon(u^\epsilon)\mathcal{T}_\epsilon(v^\epsilon)\Big\} dx dy_1 dy_2$$

$$= \int_{(0,1)\times Y^*} \frac{1}{l([x]_\epsilon)} \mathcal{T}_\epsilon(f^\epsilon)\mathcal{T}_\epsilon(v^\epsilon) dx dy_1 dy_2. \tag{2.4.16}$$

Now using the three statements in (2.4.15) and (2.4.13), we pass to the limit in (2.4.16) and we obtain the second equation:

$$\begin{aligned} &\int_W \Big(\frac{1}{l(x)}\frac{\partial u}{\partial x}(x) + \frac{\partial u_1}{\partial y_1}(x,y_1,y_2)\Big)\frac{\partial \psi}{\partial y_1}(x,y_1,y_2)\, dx dy_1 dy_2 \\ &+ \int_W \frac{\partial u_1}{\partial y_2}(x,y_1,y_2)\frac{\partial \psi}{\partial y_2}(x,y_1,y_2)\, dx dy_1 dy_2 = 0,\ \forall \psi \in C^\infty_\#(W). \end{aligned} \tag{2.4.17}$$

By density, (2.4.17) holds true for any function $\psi \in L^2\Big((0,1); H^1_\#\big(Y^*(x)\big)\Big)$. Therefore, by summing (2.4.14) and (2.4.17) we have the homogenized system (2.4.5). By a standard argument, see e.g. the proof of Theorem 1.2.4, it is easily seen that (2.4.5) satisfies the conditions of the Lax-Milgram Theorem in $H^1(0,1) \times L^2\Big((0,1); H^1_\#\big(Y^*(x)\big)/\mathbb{R}\Big)$.

To end the proof we will see the relation between the homogenized system and the classical homogenized equation (2.4.6). This is achieved by using the solutions of the problem (2.4.9). Treating x as a parameter in (2.4.17) is easy to check that it is a variational formulation associated to the following cell-problem:

$$\begin{cases} -\Delta u_1(x) = 0 \text{ in } Y^*(x) \\ \dfrac{\partial u_1(x)}{\partial N} = 0 \text{ on } B_2(x) \\ \dfrac{\partial u_1(x)}{\partial N} = \dfrac{-N_1(x)}{l(x)}\dfrac{\partial u}{\partial x}(x) \text{ on } B_1(x) \\ u_1(x,\cdot,\cdot)\ l(x) - \text{periodic in the variable } y_1, \end{cases} \tag{2.4.18}$$

where $N(x) = (N_1(x), N_2(x))$ is the unit outward normal to $\partial Y^*(x)$, $B_1(x)$ is the upper boundary and $B_2(x)$ is the lower boundary of $\partial Y^*(x)$ for all $x \in I$. Thus, taking into account u is independent of (y_1, y_2) one can see immediately that:

$$u_1(x,y_1,y_2) = -X(x)(y_1,y_2)\frac{1}{l(x)}\frac{\partial u}{\partial x}(x)\ (x,y_1,y_2) \in W, \tag{2.4.19}$$

where $X(x,\cdot,\cdot)$ is the solution of (2.4.9).

Replacing u_1 by its value, (2.4.19), in the equation (2.4.14) we obtain the weak formulation of (2.4.6).

The uniqueness and existence of weak solution of the problem (2.4.6) is an immediate consequence of the Lax-Milgram theorem.

□

Remark 2.4.2. *If the non homogeneous term $f^\epsilon(x,y)$ is a fixed function depending only on the first variable, that is, $f^\epsilon(x,y) = f(x)$, it is easy to see that $f_0(x) =$*

$\frac{|Y^*(x)|}{l(x)} f(x)$ *and therefore,* (2.4.6) *can be written as*

$$\begin{cases} -\dfrac{l(x)}{|Y^*(x)|}\big(r(x)u_x\big)_x + u = f, \quad x \in (0,1) \\ \qquad u'(0) = u'(1) = 0 \end{cases} \tag{2.4.20}$$

Remark 2.4.3. *Notice that in case G_ϵ presents a purely periodic behavior we recover the homogenized limit problem obtained in [8]. On the other hand, in case the amplitude of the oscillation depends on x but the period is constant, $l(x) \equiv L$, we obtain the same homogenized limit problem as in [9].*

Remark 2.4.4. *Observe that problem* (2.4.6) *is well posed in the sense that the diffusion coefficient $r(\cdot)$ is uniformly positive and smooth in $[0,1]$. To see that $r(\cdot)$ is positive we use the bilinear form $a(\cdot,\cdot)$ associated with the variational formulation of* (2.4.9)

$$a(\Psi,\Phi) = \int_{Y^*(x)} \nabla\Psi \cdot \nabla\Phi \, dy_1 dy_2,$$

for $\Psi, \Phi \in H^1_\#(Y^*(x))$. Then, X satisfies

$$a(X,\Phi) = \int_{B_1(x)} N_1(x)\Phi \, dS, \text{ for any } \Phi \in H^1_\#(Y^*(x)).$$

Consequently,

$$a(y_1 - X, \Phi) = \int_{B_1(x)} N_1(x)\Phi \, dS - \int_{B_1(x)} N_1(x)\Phi \, dS = 0, \text{ for any } \Phi \in H^1_\#(Y^*(x)). \tag{2.4.21}$$

Turning back to $r(\cdot)$ we have

$$\begin{aligned} r(x) &= \frac{1}{l(x)} \int_{Y^*(x)} \Big\{1 - \frac{\partial X(x,y_1,y_2)}{\partial y_1}\Big\} dy_1 dy_2 \\ &= \frac{1}{l(x)} \int_{Y^*(x)} \frac{\partial}{\partial y_1}\Big(y_1 - X(x,y_1,y_2)\Big)\frac{\partial y_1}{\partial y_1} dy_1 dy_2 = \frac{1}{l(x)} a(y_1 - X, y_1). \end{aligned} \tag{2.4.22}$$

Hence, using (2.4.22) *and* (2.4.21) *with $\Phi = -X$ we get*

$$l(x)r(x) = a(y_1 - X, y_1) - a(y_1 - X, -X) = a(y_1 - X, y_1 - X) = \|\nabla(y_1 - X)\|^2_{[L^2(Y^*(x))]^2}.$$

Therefore, since $l(x) > 0$ for all $x \in (0,1)$ we can conclude that $r(\cdot) > 0$. Indeed, if this is not true, then we would have

$$\frac{\partial(y_1 - X)}{\partial y_1} = 0, \quad \frac{\partial(y_1 - X)}{\partial y_2} = 0$$

which implies that there exists a constant C such that $y_1 - X(x,\cdot,\cdot) = C$ for each $x \in (0,1)$. This means that $X(x,y_1,y_2) = y_1 - C$ which is impossible since $X(x,\cdot,\cdot)$ is $l(x)$-periodic in the variable y_1.

We discuss now the smoothness of the diffusion coefficient $r(\cdot)$. *Notice that from its definition in* (2.4.8), *its smoothness is directly related to the smoothness of the function* $x \to l(x)$, $x \to |Y^*(x)|$ *and* $x \to X(x, \cdot, \cdot)$. *Observe that, as* x *moves in the interval* $(0, 1)$, *the domain* $Y^*(x)$ *varies changing its height and width in a smooth manner. This variation of the domain affects the function* X *since it is defined as the solution of* (2.4.9). *Also, notice that from standard elliptic regularity theory, for fixed* $x \in Y^*(x)$ *the function* $X(x, \cdot, \cdot) \in C^1_\#\big(Y^*(x)\big)$, *see for instance [67].*

Moreover, since both functions $l(x)$ *and* $G(x, y)$ *are smooth, then the domain perturbation* $x \to Y^*(x)$ *is also smooth and this will imply that the function* $x \to X(x, \cdot, \cdot) \in C^1_\#\big(Y^*(x)\big)$ *is smooth.*

The standard way to prove this is to construct a family of diffeomorphisms L_x *which transform the cell* $Y^*(x)$ *into a fixed cell* Z^*. *This map* L_x *will allow us to transform problem* (2.4.9) *into a problem in the fixed cell* Z^* *but where the differential equation and boundary conditions have coefficients depending on* x. *This coefficients are as smooth as the smoothness of the diffeomorphisms in terms of* x. *Thus, in our particular case we may consider*

$$\begin{aligned} L_x \ &: \ Y^*(x) \mapsto Z^* \\ &\quad (y_1, y_2) \to \Big(\frac{y_1}{l(x)}, \hat{F}(y_1)\, y_2\Big) = (z_1, z_2) \end{aligned}$$

where $\hat{F}(y_1) = \frac{\hat{G}(y_1/l(x))}{G(x,y_1)}$, $\hat{G} \in \mathcal{C}^1(\mathbb{R})$ *is a* $1-$*periodic function and*

$$Z^* = \{(z_1, z_2) \in \mathbb{R}^2 \ : \ 0 < z_1 < 1, \quad 0 < z_2 < \hat{G}(z_1)\}.$$

Then, diffeomorphisms L_x *allow us to transform the auxiliary problems* (2.4.9) *into equivalent problems in the fixed cell* Z^* *and show that the solution of the transformed problem is smooth enough, say* $C^1([0, 1], C^1_\#\big(Z^*)\big)$. *Undoing the transformation we get that* $X \in C^1\big([0, 1]; C^1_\#\big(Y^*(x)\big)\big)$. *A very similar reasoning was used in [97, Proposition A.2], or [9, Proposition A.1]. We also refer to [74, Chapter 2] for more general results in domain perturbation problems and smoothness of solutions with respect to variation of the domain.*

Therefore, we have that the diffusion coefficient $r(\cdot)$ *is uniformly positive and* C^1 *in* $[0, 1]$ *which implies that the solution of the homogenized problem* (2.4.6) *satisfies* $u \in H^2(0, 1) \cap C^1(0, 1)$.

2.5. Corrector result for the Neumann problem

In this section we address the question of correctors for problem (2.0.1). To do that, we need the averaging operator $\mathcal{U}_\epsilon$, adapted to locally periodic thin domains, see Section 1.1 of Chapter 1 for the definition for the purely periodic case. In principle, this operator could be associated to any "admissible partition" $\{x^\epsilon_k\}$ but since we will use it in connection with convergence properties of the solutions and will use the results from previous sections, we will consider it is already associated to the $l(x)$-partition, see Definition 2.3.1. Hence, we define

Definition 2.5.1. *Let $\{x_k^\epsilon\}$ be the $l(x)$-partition. Then, if $\varphi \in L^p((0,1)\times Y^*)$, $p \in [1,\infty]$, we set*

$$\mathcal{U}_\epsilon(\varphi)(x,y) = \frac{1}{l([x]_\epsilon)} \int_0^{l([x]_\epsilon)} \varphi\Big([x]_\epsilon + \Gamma_{[x]_\epsilon} y_1, \frac{x-[x]_\epsilon}{\Gamma_{[x]_\epsilon}}, \frac{y}{\epsilon}\Big)\, dy_1, \quad \forall (x,y) \in R^\epsilon.$$

Proposition 2.5.2. *The main properties of $\mathcal{U}_\epsilon$ are the following:*

i) Assume $1 \le p,q \le \infty$ and $\frac{1}{p}+\frac{1}{q}=1$. The averaging operator $\mathcal{U}_\epsilon$ is the formal adjoint of the unfolding operator $\mathcal{T}_\epsilon$, in the sense that

$$\int_{(0,1)\times Y^*} \frac{1}{l([x]_\epsilon)} \mathcal{T}_\epsilon(\varphi)\psi\, dxdy_1dy_2 = \frac{1}{\epsilon}\int_{R^\epsilon} \varphi \mathcal{U}_\epsilon(\psi)\, dxdy,$$

$\forall \varphi \in L^q(R^\epsilon)$ and $\psi \in L^p((0,1)\times Y^*)$.

ii) Let p belong to $[1,\infty]$. The averaging operator $\mathcal{U}_\epsilon$ is linear continuous from $L^p((0,1)\times Y^*)$ to $L^p(R^\epsilon)$ and there exists a constant $C>0$ independent of ϵ such that for $1 \le p < \infty$

$$|||\mathcal{U}_\epsilon(\varphi)|||_{L^p(R^\epsilon)} \leqslant C\|\varphi\|_{L^p\big((0,1)\times Y^*\big)}, \quad \forall \varphi \in L^p((0,1)\times Y^*),$$

and for $p=\infty$

$$\|\mathcal{U}_\epsilon(\varphi)\|_{L^\infty(R^\epsilon)} \leqslant C\|\varphi\|_{L^\infty\big((0,1)\times Y^*\big)}, \quad \forall \varphi \in L^\infty((0,1)\times Y^*).$$

iii) $\mathcal{U}_\epsilon$ is the left inverse of $\mathcal{T}_\epsilon$, that is $(\mathcal{U}_\epsilon \circ \mathcal{T}_\epsilon)(\phi) = \phi$ for every $\phi \in L^p(R^\epsilon)$ with $1 \le p \le \infty$.

iv) Suppose that $p \in [1,\infty)$. Let $\varphi \in L^p(0,1)$. Then, $|||\mathcal{U}_\epsilon(\varphi)-\varphi|||_{L^p(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0$.

v) Let $\{\varphi^\epsilon\}$ be a sequence in $L^p(R^\epsilon)$, $p \in [1,\infty)$, such that

$$\mathcal{T}_\epsilon(\varphi^\epsilon) \overset{\epsilon\to 0}{\longrightarrow} \varphi \ s - L^p\big((0,1)\times Y^*\big).$$

Then

$$|||\mathcal{U}_\epsilon(\varphi)-\varphi^\epsilon|||_{L^p(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0.$$

Proof. i) For every $\varphi \in L^q(R^\epsilon)$ and $\psi \in L^p((0,1)\times Y^*)$ we have

$$\int_{(0,1)\times Y^*} \frac{1}{l([x]_\epsilon)} \mathcal{T}_\epsilon(\varphi)\psi\, dxdy_1dy_2 =$$

$$= \sum_{k=0}^{N_\epsilon} \int_{(x_k^\epsilon, x_{k+1}^\epsilon)\times Y^*} \frac{1}{l(x_k^\epsilon)} \widetilde{\varphi}\Big(x_k^\epsilon + \Gamma_{x_k^\epsilon} y_1, \epsilon y_2\Big) \chi_{(0,l(x_k^\epsilon))}(y_1)\psi\, dxdy_1dy_2$$

$$= \sum_{k=0}^{N_\epsilon} \int_{(0,l(x_k^\epsilon))\times Y^*} \frac{1}{l(x_k^\epsilon)} \widetilde{\varphi}\Big(x_k^\epsilon + \Gamma_{x_k^\epsilon} y_1, \epsilon y_2\Big) \chi_{(0,l(x_k^\epsilon))}(y_1)\psi\Big(x_k^\epsilon + \Gamma_{x_k^\epsilon} z, y_1, y_2\Big)\Gamma_{x_k^\epsilon}\, dzdy_1dy_2$$

$$= \sum_{k=0}^{N_\epsilon} \int_{(0,l(x_k^\epsilon))\times(x_k^\epsilon,x_{k+1}^\epsilon)\times(0,G_1)} \frac{1}{l(x_k^\epsilon)} \widetilde{\varphi}(x,\epsilon y_2)\psi\Big(x_k^\epsilon + \Gamma_{x_k^\epsilon} z, \frac{x-x_k^\epsilon}{\Gamma_{x_k^\epsilon}}, y_2\Big)\, dzdxdy_2$$
$$= \sum_{k=0}^{N_\epsilon} \frac{1}{\epsilon} \int_{(x_k^\epsilon,x_{k+1}^\epsilon)\times(0,\epsilon G_1)} \widetilde{\varphi}(x,y)\Big(\frac{1}{l(x_k^\epsilon)} \int_{(0,l(x_k^\epsilon))} \psi\Big(x_k^\epsilon + \Gamma_{x_k^\epsilon} z, \frac{x-x_k^\epsilon}{\Gamma_{x_k^\epsilon}}, \frac{y}{\epsilon}\Big)\, dz\Big)\, dxdy$$
$$= \frac{1}{\epsilon}\int_{R^\epsilon} \varphi\mathcal{U}_\epsilon(\psi)\, dxdy.$$

ii) For $p=1$, it is a immediate consequence of the duality above. For $p>1$, from the duality above and of the property iv) in Proposition 2.2.5 we have

$$\begin{aligned}
|||\mathcal{U}_\epsilon(\varphi)|||^p_{L^p(R^\epsilon)} &= \frac{1}{\epsilon}\int_{R^\epsilon} |\mathcal{U}_\epsilon(\varphi)^{p-1}\mathcal{U}_\epsilon(\varphi)|\, dxdy \\
&= \int_{(0,1)\times Y^*} \frac{1}{l([x]_\epsilon)} \big|\mathcal{T}_\epsilon\big(\mathcal{U}_\epsilon(\varphi)\big)^{p-1}\varphi\big|\, dxdy_1dy_2 \\
&\leq \frac{1}{l_1} ||\mathcal{T}_\epsilon\big(\mathcal{U}_\epsilon(\varphi)\big)^{p-1}||_{L^{\frac{p}{p-1}}\big((0,1)\times Y^*\big)} ||\varphi||_{L^p\big((0,1)\times Y^*\big)} \\
&\leq \Big(\frac{l_1}{\epsilon}\Big)^{\frac{p}{p-1}} \frac{1}{l_1} ||\mathcal{U}_\epsilon(\varphi)^{p-1})||_{L^{\frac{p}{p-1}}(R^\epsilon)} ||\varphi||_{L^p\big((0,1)\times Y^*\big)}
\end{aligned}$$

Therefore, since $\dfrac{p-1}{p} + \dfrac{1}{p} = 1$ we obtain

$$|||\mathcal{U}_\epsilon(\varphi)|||_{L^p(R^\epsilon)} \leqslant \Big(\frac{1}{l_1}\Big)^{1/p} \|\varphi\|_{L^p\big((0,1)\times Y^*\big)}.$$

For $p=\infty$ the statement is satisfied by the definition of the $\mathcal{U}_\epsilon(\varphi)$ and since $0 < l_0 \leqslant l(x) < l_1$

$$||\mathcal{U}_\epsilon(\varphi)||_{L^\infty(R^\epsilon)} \leq \frac{1}{l_0}\int_0^{l_1} ||\varphi||_{L^\infty(W)} \leq C||\varphi||_{L^\infty(W)}.$$

iii) Simple consequence of the definition of the operator $\mathcal{U}_\epsilon$.

iv) The result is clear for any $\varphi \in \mathcal{D}(0,1)$. By density, we obtain the convergence.

v) It is a direct consequence of the properties ii) and iii). Observe that

$$|||\mathcal{U}_\epsilon(\varphi) - \varphi^\epsilon|||_{L^p(R^\epsilon)} = |||\mathcal{U}_\epsilon\big(\varphi - \mathcal{T}_\epsilon(\varphi^\epsilon)\big)|||_{L^p(R^\epsilon)} \leqslant C\|\varphi - \mathcal{T}_\epsilon(\varphi^\epsilon)\|_{L^p\big((0,1)\times Y^*\big)}.$$

□

Finally, we give a general corrector result. We show convergence in H^1-norms if we add the first-order corrector term, $\epsilon\frac{\partial u}{\partial x}X(x,x/\epsilon,y/\epsilon)$, to the original solutions u^ϵ. In order to simplify the notation we will use X^ϵ to denote $X(x,x/\epsilon,y/\epsilon)$, that is, $X^\epsilon(x,y) \equiv X(x,x/\epsilon,y/\epsilon)$ for all $(x,y)\in R^\epsilon$. Recall that the function X is the auxiliary function introduced by problem (2.4.9) for every $x\in(0,1)$.

Theorem 2.5.3. *Assume hypotheses of Theorem 2.4.1 hold. Then,*

i) $\lim_{\epsilon\to 0} |||u^\epsilon - u|||_{L^2(R^\epsilon)} = 0.$

ii) $\lim_{\epsilon\to 0} ||\mathcal{T}_\epsilon(\nabla u^\epsilon) - \big(\nabla u + l(\cdot)(\nabla_{y_1y_2}u_1)\big)\chi||_{\left[L^2\left((0,1)\times Y^*\right)\right]^2} = 0.$

iii) $\lim_{\epsilon\to 0} |||\nabla u^\epsilon - \nabla u - \mathcal{U}_\epsilon(l(\cdot)(\nabla_{y_1y_2}u_1)\chi)|||_{[L^2(R^\epsilon)]^2} = 0.$

iv) $\lim_{\epsilon\to 0} \left|\left|\left|u^\epsilon - u + \epsilon\dfrac{\partial u}{\partial x}X^\epsilon\right|\right|\right|_{H^1(R^\epsilon)} = 0.$

Remark 2.5.4. *Notice that the special feature of the first-order corrector function,* $\epsilon\frac{\partial u}{\partial x}X(x, x/\epsilon, y/\epsilon)$*, is that the solution* $X(x,\cdot,\cdot)$ *to the auxiliary problem depends on* x*. This dependence on* x *is a consequence of the locally periodic behavior of the thin domain.*

Proof of Th. 6.3. i) Performing the change of variables $(x, y) \to (x, y/\epsilon)$ we obtain the following equivalent linear elliptic problem to (2.0.1)

$$\begin{cases} -\dfrac{\partial^2 w^\epsilon}{\partial x^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 w^\epsilon}{\partial y^2} + w^\epsilon = f^\epsilon & \text{in } \Omega^\epsilon, \\ \dfrac{\partial w^\epsilon}{\partial x}N_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial w^\epsilon}{\partial y}N_2^\epsilon = 0 & \text{on } \partial\Omega^\epsilon, \end{cases}$$

where $f^\epsilon \in L^2(\Omega^\epsilon)$ is uniformly bounded, $N^\epsilon = (N_1^\epsilon, N_2^\epsilon)$ is the outward unit normal to $\partial\Omega^\epsilon$ and Ω^ϵ is given by

$$\Omega^\epsilon = \{(x, y) \in \mathbb{R}^2 \mid x \in (0, 1),\ 0 < y < G(x, x/\epsilon)\}.$$

Observe that w^ϵ satisfies the following a priori estimates

$$\|w^\epsilon\|_{L^2(\Omega^\epsilon)},\ \left\|\frac{\partial w^\epsilon}{\partial x}\right\|_{L^2(\Omega^\epsilon)},\ \frac{1}{\epsilon}\left\|\frac{\partial w^\epsilon}{\partial y}\right\|_{L^2(\Omega^\epsilon)} \le C. \tag{2.5.1}$$

Now, using a reflection in the vertical direction of the oscillatory boundary, see [8], it is not difficult to define an extension operator $P^\epsilon : H^1(\Omega^\epsilon) \to H^1(\Omega)$, $\Omega = (0, 1) \times (0, G_1)$, verifying

$$\begin{aligned} &||P_\epsilon w^\epsilon||_{L^2(\Omega)} \le C||w^\epsilon||_{L^2(\Omega^\epsilon)}, \\ &\left\|\frac{\partial P_\epsilon w^\epsilon}{\partial x}\right\|_{L^2(\Omega)} \le C\left\{\left\|\frac{\partial w^\epsilon}{\partial x}\right\|_{L^2(\Omega^\epsilon)} + \frac{1}{\epsilon}\left\|\frac{\partial w^\epsilon}{\partial y}\right\|_{L^2(\Omega^\epsilon)}\right\}, \\ &\left\|\frac{\partial P_\epsilon w^\epsilon}{\partial y}\right\|_{L^2(\Omega)} \le C\left\|\frac{\partial w^\epsilon}{\partial y}\right\|_{L^2(\Omega^\epsilon)}. \end{aligned}$$

Then, from the a priori estimates (2.5.1) and the inequalities above it follows straightforward that there exist a subsequence of $\{P_\epsilon w^\epsilon\}$, denoted again by $\{P_\epsilon w^\epsilon\}$, and $w \in H^1(\Omega)$ such that

$$P_\epsilon w^\epsilon \overset{\epsilon\to 0}{\rightharpoonup} w \quad w - H^1(\Omega).$$
$$\frac{\partial P_\epsilon w^\epsilon}{\partial y} \overset{\epsilon\to 0}{\longrightarrow} 0 \quad s - L^2(\Omega).$$

Consequently, $\|w^\epsilon - w\|_{L^2(\Omega^\epsilon)} \to 0$ as $\epsilon \to 0$ and since $\left\|\frac{\partial P_\epsilon w^\epsilon}{\partial y}\right\| \to 0$ we have that w does not depend on y, that is, $w \in H^1(0,1)$.

Undoing the change of variables we get

$$|||u^\epsilon - w|||_{L^2(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0. \tag{2.5.2}$$

Finally, we have to see that $w = u$. Then, from (2.5.2) and using property iv) of Proposition 1.1.4 and Proposition 1.1.10 we have

$$\lim_{\epsilon\to 0} ||\mathcal{T}_\epsilon(u^\epsilon) - w||_{L^2\big((0,1)\times Y^*\big)} = 0.$$

Therefore, taking into account convergence (2.4.2) and the uniqueness of the limit we obtain that $w = u$.

ii) This convergence improves the convergences (2.4.3) and (2.4.4). It is based on the convergence of the energy. Taking $\varphi = \frac{u^\epsilon}{l(x)}$ in the variational formulation (2.4.1), we obtain

$$\int_{R^\epsilon} \Big(\frac{1}{l(x)} \Big(\frac{\partial u^\epsilon}{\partial x}\Big)^2 - u_\epsilon \frac{\partial u^\epsilon}{\partial x} \frac{l'(x)}{l(x)^2} + \frac{1}{l(x)} \Big(\frac{\partial u^\epsilon}{\partial y}\Big)^2 \Big)\, dxdy$$
$$= \int_{R^\epsilon} \Big(\frac{f^\epsilon u^\epsilon}{l(x)} - \frac{(u^\epsilon)^2}{l(x)} \Big)\, dxdy.$$

Therefore, using the unfolding criterion for integrals and passing to the limit we get the following convergence

$$\int_{(0,1)\times Y^*} \frac{1}{l(x)^2} \Big\{ \mathcal{T}_\epsilon \Big(\frac{\partial u^\epsilon}{\partial x}\Big)^2 + \mathcal{T}_\epsilon \Big(\frac{\partial u^\epsilon}{\partial y}\Big)^2 \Big\}\, dxdy_1dy_2$$
$$\overset{\epsilon\to 0}{\longrightarrow} \int_W \left\{ \frac{\hat{f}u - u^2}{l(x)^2} + \frac{1}{l(x)^2}\Big(u\frac{\partial u}{\partial x}\frac{l'(x)}{l(x)} + u\frac{\partial u_1}{\partial y_1} l'(x) \Big) \right\} dxdy_1dy_2. \tag{2.5.3}$$

On the other hand, choosing $\phi = \frac{u}{l(x)}$ and $\psi = u_1$ as test functions in (2.4.5) we get

$$\int_W \left\{ \Big(\frac{1}{l(x)}\frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial y_1} \Big)^2 + \frac{\partial u_1}{\partial y_2}^2 \right\} dxdy_1dy_2$$
$$= \int_W \left\{ \frac{\hat{f}u - u^2}{l(x)^2} + \frac{1}{l(x)^2}\Big(u\frac{\partial u}{\partial x}\frac{l'(x)}{l(x)} + u\frac{\partial u_1}{\partial y_1} l'(x) \Big) \right\} dxdy_1dy_2. \tag{2.5.4}$$

Then, combining (2.5.3) and (2.5.4) we have

$$\begin{aligned}
&\int_{(0,1)\times Y^*} \frac{1}{l(x)^2} \left\{ \mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x}\Big)^2 + \mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial y}\Big)^2 \right\} dx dy_1 dy_2 \\
&\quad\stackrel{\epsilon\to 0}{\longrightarrow} \int_W \left\{ \Big(\frac{1}{l(x)}\frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial y_1}\Big)^2 + \frac{\partial u_1}{\partial y_2}^2 \right\} dx dy_1 dy_2. \qquad (2.5.5)
\end{aligned}$$

Consequently, due to the weak convergences (2.4.3), (2.4.4) and the convergence (2.5.5) we can ensure by the Radon-Riesz property the strong convergence *ii)*.

iii) From strong convergence ii) and using property ii) of Proposition 2.5.2 one immediately has

$$\lim_{\epsilon\to 0} |||\mathcal{U}_\epsilon\big(\mathcal{T}_\epsilon(\nabla u^\epsilon)\big) - \mathcal{U}_\epsilon((\nabla u)\chi) - \mathcal{U}_\epsilon\big(l(x)(\nabla_{y_1 y_2} u_1)\chi\big)|||_{[L^2(R^\epsilon)]^2} = 0. \quad (2.5.6)$$

But from Proposition 1.1.10 we have

$$\lim_{\epsilon\to 0} ||\mathcal{T}_\epsilon(\nabla u) - \nabla u\,\chi||_{\left[L^2\left((0,1)\times Y^*\right)\right]^2} = 0,$$

which using property v) of Proposition 2.5.2 implies

$$|||\mathcal{U}_\epsilon((\nabla u)\chi) - \nabla u|||_{[L^2(R^\epsilon)]^2} \stackrel{\epsilon\to 0}{\longrightarrow} 0. \qquad (2.5.7)$$

Now, by the triangular inequality and taking into account that $\mathcal{U}_\epsilon$ is the left inverse of $\mathcal{T}_\epsilon$ we obtain

$$\begin{aligned}
|||\nabla u^\epsilon - \nabla u - \mathcal{U}_\epsilon(l(x)(\nabla_{y_1 y_2} u_1)\chi)|||_{[L^2(R^\epsilon)]^2} &\le |||\mathcal{U}_\epsilon((\nabla u)\chi) - \nabla u|||_{[L^2(R^\epsilon)]^2} \\
&+ |||\mathcal{U}_\epsilon\big(\mathcal{T}_\epsilon(\nabla u^\epsilon)\big) - \mathcal{U}_\epsilon((\nabla u)\chi) - \mathcal{U}_\epsilon(l(x)(\nabla_{y_1 y_2} u_1)\chi)|||_{[L^2(R^\epsilon)]^2}.
\end{aligned}$$

Then, from (2.5.6) and (2.5.7) we obtain the desired convergence.

iv) Recall that due to the regularity of the functions $G(\cdot,\cdot)$ and $l(\cdot)$ we can ensure that the function X belongs, at least, to $C^1\big([0,1]; C^1_\#\big(Y^*(x)\big)\big)$, see Remark 2.4.4. Then, the function X^ϵ is a well-defined function in $H^1(R^\epsilon)$ and we can obtain some estimates in R^ϵ. It is easy to see that

$$\begin{aligned}
|||X^\epsilon|||^2_{L^2(R^\epsilon)} &= \frac{1}{\epsilon}\int_{R^\epsilon} |X(x, x/\epsilon, y/\epsilon)|^2\, dx dy \\
&= \int_{(0,1)} \int_{(0,G(x,x/\epsilon))} |X(x, x/\epsilon, z)|^2\, dx dz \\
&\le C \int_{(0,1)} \sup_{(y_1,y_2)\in Y^*(x)} |X(x, y_1, y_2)|^2\, dx \\
&= C||X||^2_{L^2\left((0,1); L^\infty_\#\left(Y^*(x)\right)\right)}, \qquad (2.5.8)
\end{aligned}$$

where we use that $X(x,\cdot,\cdot)$ is periodic in the variable y_1.

Moreover, since $X \in C^1\big([0,1]; C^1_{\#}\big(Y^*(x)\big)\big)$ it follows that $X \in L^\infty(W)$ and we have

$$||X^\epsilon||_{L^\infty(R^\epsilon)} \leq ||X||_{L^\infty(W)}. \tag{2.5.9}$$

In order to simplify the notation we consider the following functions,

$$X_0^\epsilon(x,y) \equiv \frac{\partial X}{\partial x}\big(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}\big), \quad X_1^\epsilon(x,y) \equiv \frac{\partial X}{\partial y_1}\big(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}\big),$$

and $X_2^\epsilon(x,y) \equiv \dfrac{\partial X}{\partial y_2}\big(x, \frac{x}{\epsilon}, \frac{y}{\epsilon}\big)$, $\forall (x,y) \in R^\epsilon$.

Analogously to (2.5.8), we can get

$$|||X_0^\epsilon(x,y)|||^2_{L^2(R^\epsilon)} \leq C\Big|\Big|\frac{\partial X}{\partial x}\Big|\Big|^2_{L^2\left((0,1);L^\infty_{\#}\left(Y^*(x)\right)\right)}. \tag{2.5.10}$$

Notice that since $X \in C^1\big([0,1]; C^1_{\#}\big(Y^*(x)\big)\big)$ we can ensure that $\frac{\partial X}{\partial y_1}, \frac{\partial X}{\partial y_2} \in C^0_{\#}(W)$.

By the definition of the norm $|||\cdot|||_{H^1(R^\epsilon)}$ we have

$$\begin{aligned}
\Big|\Big|\Big|u^\epsilon - u + \epsilon\frac{\partial u}{\partial x}X^\epsilon\Big|\Big|\Big|^2_{H^1(R^\epsilon)} &= \Big|\Big|\Big|u^\epsilon - u + \epsilon\frac{\partial u}{\partial x}X^\epsilon\Big|\Big|\Big|^2_{L^2(R^\epsilon)} \\
&+ \Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial x} - \frac{\partial u}{\partial x} + \epsilon\frac{\partial u}{\partial x}X_0^\epsilon + \frac{\partial u}{\partial x}X_1^\epsilon + \epsilon\frac{\partial^2 u}{\partial x^2}X^\epsilon\Big|\Big|\Big|^2_{L^2(R^\epsilon)} \\
&+ \Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial y} + \frac{\partial u}{\partial x}X_2^\epsilon\Big|\Big|\Big|^2_{L^2(R^\epsilon)}.
\end{aligned} \tag{2.5.11}$$

Note that the corrector is well defined due to the smoothness of the function X and since $u \in H^2(0,1) \cap C^1(0,1)$.

Now, we calculate the limit for each term of (2.5.11). For the first term we have the following inequality:

$$\Big|\Big|\Big|u^\epsilon - u + \epsilon\frac{\partial u}{\partial x}X^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)} \leqslant |||u^\epsilon - u|||_{L^2(R^\epsilon)} + \Big|\Big|\Big|\epsilon\frac{\partial u}{\partial x}X^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)}. \tag{2.5.12}$$

Observe that, due to estimate (2.5.8) we get

$$\Big|\Big|\Big|\epsilon\frac{\partial u}{\partial x}X^\epsilon\Big|\Big|_{L^2(R^\epsilon)} \leq \epsilon\Big|\Big|\frac{\partial u}{\partial x}\Big|\Big|_{L^\infty(0,1)}|||X^\epsilon|||_{L^2(R^\epsilon)} \leq C\epsilon. \tag{2.5.13}$$

Therefore, from convergence i), estimate (2.5.13) and inequality (2.5.12) we obtain the convergence of the first term

$$\Big|\Big|\Big|u^\epsilon - u + \epsilon\frac{\partial u}{\partial x}X^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0. \tag{2.5.14}$$

For the second term, adding and subtracting the appropriate functions and with the triangular inequality we obtain

$$\Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial x} - \frac{\partial u}{\partial x} + \epsilon\frac{\partial u}{\partial x}X_0^\epsilon + \frac{\partial u}{\partial x}X_1^\epsilon + \epsilon\frac{\partial^2 u}{\partial x^2}X^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)} \le I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &= \Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial x} - \frac{\partial u}{\partial x} - \mathcal{U}_\epsilon(l(x)\frac{\partial u_1}{\partial y_1}\chi)\Big|\Big|\Big|_{L^2(R^\epsilon)},\\
I_2 &= \Big|\Big|\Big|\mathcal{U}_\epsilon(l(x)\frac{\partial u_1}{\partial y_1}\chi) + \frac{\partial u}{\partial x}X_1^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)},\\
I_3 &= \Big|\Big|\Big|\epsilon\frac{\partial u}{\partial x}X_0^\epsilon + \epsilon\frac{\partial^2 u}{\partial x^2}X^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)}.
\end{aligned}$$

It follows from convergence iii) that

$$I_1 = \Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial x} - \frac{\partial u}{\partial x} - \mathcal{U}_\epsilon\big(l(x)\frac{\partial u_1}{\partial y_1}\chi\big)\Big|\Big|\Big|_{L^2(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0.$$

Since $\mathcal{T}_\epsilon\big(\frac{\partial u}{\partial x}X_1^\epsilon\big) = \mathcal{T}_\epsilon\big(\frac{\partial u}{\partial x}\big)\mathcal{T}_\epsilon(X_1^\epsilon)$ and taking into account that by Proposition 1.1.10 we have

$$\mathcal{T}_\epsilon\Big(\frac{\partial u}{\partial x}\Big) \overset{\epsilon\to 0}{\longrightarrow} \frac{\partial u}{\partial x}\chi \quad \mathrm{s} - L^2\big((0,1)\times Y^*\big),$$

and using the same arguments as Proposition 2.3.7 it follows

$$\mathcal{T}_\epsilon(X_1^\epsilon) \overset{\epsilon\to 0}{\longrightarrow} \frac{\partial X}{\partial y_1}\chi \quad \mathrm{s} - L^2\big((0,1)\times Y^*\big),$$

then, we obtain

$$\mathcal{T}_\epsilon\Big(\frac{\partial u}{\partial x}X_1^\epsilon\Big) \overset{\epsilon\to 0}{\longrightarrow} \frac{\partial u}{\partial x}\frac{\partial X}{\partial y_1}\chi \quad \mathrm{s} - L^2\big((0,1)\times Y^*\big).$$

Consequently, from property v) in Proposition 2.5.2 we obtain

$$\Big|\Big|\Big|\frac{\partial u}{\partial x}X_1^\epsilon - \mathcal{U}_\epsilon\Big(\frac{\partial u}{\partial x}\frac{\partial X}{\partial y_1}\chi\Big)\Big|\Big|\Big|_{L^2(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0.$$

Therefore, since from (2.4.19) one has $-\mathcal{U}_\epsilon(\frac{\partial u}{\partial x}\frac{\partial X}{\partial y_1}\chi) = \mathcal{U}_\epsilon(l(x)\frac{\partial u_1}{\partial y_1}\chi)$ we obtain

$$I_2 = \Big|\Big|\Big|\mathcal{U}_\epsilon\Big(l(x)\frac{\partial u_1}{\partial y_1}\Big) + \frac{\partial u}{\partial x}X_1^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0.$$

Using the triangular inequality we have for I_3

$$\Big|\Big|\Big|\epsilon\frac{\partial u}{\partial x}X_0^\epsilon+\epsilon\frac{\partial^2 u}{\partial x^2}X^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)}\le\Big|\Big|\Big|\epsilon\frac{\partial u}{\partial x}X_0^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)}+\Big|\Big|\Big|\epsilon\frac{\partial^2 u}{\partial x^2}X^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)} \tag{2.5.15}$$

Moreover, in view of (2.5.10) and (2.5.9) and taking into account that $u\in H^2(0,1)\cap C^1(0,1)$ we obtain

$$\Big|\Big|\Big|\epsilon\frac{\partial u}{\partial x}X_0^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)}\le\epsilon\Big|\Big|\frac{\partial u}{\partial x}\Big|\Big|_{L^\infty(0,1)}|||X_0^\epsilon|||_{L^2(R^\epsilon)}\le C\epsilon \tag{2.5.16}$$

$$\Big|\Big|\Big|\epsilon\frac{\partial^2 u}{\partial x^2}X^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)}\le\epsilon||X^\epsilon||_{L^\infty(R^\epsilon)}\Big|\Big|\Big|\frac{\partial^2 u}{\partial x^2}\Big|\Big|\Big|_{L^2(R^\epsilon)}\le C\epsilon \tag{2.5.17}$$

Therefore, by (2.5.15), (2.5.16) and (2.5.17)we have

$$I_3=\Big|\Big|\Big|\epsilon\frac{\partial u}{\partial x}X_0^\epsilon+\epsilon\frac{\partial^2 u}{\partial x^2}X^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)}\overset{\epsilon\to 0}{\longrightarrow}0.$$

Then, we have proved that

$$\Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial x}-\frac{\partial u}{\partial x}+\epsilon\frac{\partial u}{\partial x}X_0^\epsilon+\frac{\partial u}{\partial x}X_1^\epsilon+\epsilon\frac{\partial^2 u}{\partial x^2}X^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)}\overset{\epsilon\to 0}{\longrightarrow}0. \tag{2.5.18}$$

Finally, arguing as for the second term, we obtain the convergence for the third term of (2.5.11). Indeed, adding and subtracting the appropriate functions and with the triangular inequality we obtain

$$\Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial y}+\frac{\partial u}{\partial x}X_2^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)}\le\Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial y}-\mathcal{U}_\epsilon(l(x)\frac{\partial u_1}{\partial y_2}\chi)\Big|\Big|\Big|_{L^2(R^\epsilon)}+\Big|\Big|\Big|\mathcal{U}_\epsilon(l(x)\frac{\partial u_1}{\partial y_2}\chi)+\frac{\partial u}{\partial x}X_2^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)}.$$

Now we pass to the limit in each term of the right-hand side of the inequality above.

On the one hand, it follows from convergence iii) that

$$\Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial y}-\mathcal{U}_\epsilon(l(x)\frac{\partial u_1}{\partial y_2}\chi)\Big|\Big|\Big|_{L^2(R^\epsilon)}\overset{\epsilon\to 0}{\longrightarrow}0.$$

On the other hand, since $\mathcal{T}_\epsilon\big(\frac{\partial u}{\partial x}X_2^\epsilon\big)=\mathcal{T}_\epsilon\big(\frac{\partial u}{\partial x}\big)\mathcal{T}_\epsilon(X_2^\epsilon)$ and taking into account Proposition 1.1.10 and Proposition 2.3.7 we have

$$\mathcal{T}_\epsilon\Big(\frac{\partial u}{\partial x}X_2^\epsilon\Big)\overset{\epsilon\to 0}{\longrightarrow}\frac{\partial u}{\partial x}\frac{\partial X}{\partial y_2}\chi\quad \mathrm{s}-L^2\big((0,1)\times Y^*\big).$$

Consequently, from property v) in Proposition 2.5.2 we obtain

$$\Big|\Big|\Big|\frac{\partial u}{\partial x}X_2^\epsilon - \mathcal{U}_\epsilon\Big(\frac{\partial u}{\partial x}\frac{\partial X}{\partial y_2}\chi\Big)\Big|\Big|\Big|_{L^2(R^\epsilon)} \xrightarrow{\epsilon\to 0} 0.$$

Thus, since $-\mathcal{U}_\epsilon(\frac{\partial u}{\partial x}\frac{\partial X}{\partial y_2}\chi) = \mathcal{U}_\epsilon(l(x)\frac{\partial u_1}{\partial y_2}\chi)$, see (2.4.19), we obtain

$$\Big|\Big|\Big|\mathcal{U}_\epsilon(l(x)\frac{\partial u_1}{\partial y_2}\chi) + \frac{\partial u}{\partial x}X_2^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)} \xrightarrow{\epsilon\to 0} 0.$$

Hence, we have proved that

$$\Big|\Big|\Big|\frac{\partial u^\epsilon}{\partial y} + \frac{\partial u}{\partial x}X_2^\epsilon\Big|\Big|\Big|_{L^2(R^\epsilon)} \xrightarrow{\epsilon\to 0} 0. \tag{2.5.19}$$

Therefore, in light of (2.5.11), (2.5.14), (2.5.18) and (2.5.19) we prove iv).

□

Chapter 3

Thin domains with doubly oscillatory boundary

We continue in this chapter the study of the behavior of the Poisson equation in thin domains going beyond the purely periodic case. In Chapter 2, we have studied the case where the profile of the thin domain is locally periodic in the sense that the oscillations at the boundary are described by a function which may change the amplitude and period with space.

In the present chapter we consider that top and bottom boundary of the thin domain oscillate with different profiles (although we will consider both of them periodic) and possibly with different order of frequency. In this respect, it does not exist a basic cell (as in the cases in Chapter 1) from which the thin domain can be obtained by appropriate rescaling and cell repetition, as it is the case of Chapter 1 (see the introduction to Chapter 1). We are still interested in understanding how the microstructure of both boundaries influence the macro properties of the problem.

Hence, we still consider the following Neumann problem

$$\begin{cases} -\Delta w^\epsilon + w^\epsilon = f^\epsilon & \text{in } R^\epsilon, \\ \dfrac{\partial w^\epsilon}{\partial \nu^\epsilon} = 0 & \text{on } \partial R^\epsilon, \end{cases} \tag{3.0.1}$$

where $f^\epsilon \in L^2(R^\epsilon)$ uniformly bounded in ϵ, $|||f^\epsilon|||_{L^2(R^\epsilon)} \le C$ (recall that the rescaled norm $|||f^\epsilon|||_{L^2(R^\epsilon)} = \epsilon^{-1/2}||f^\epsilon||_{L^2(R^\epsilon)}$ was introduced in Notation Section), and ν^ϵ is the unit outward normal to ∂R^ϵ. The domain R^ϵ is a two-dimensional thin domain which presents an oscillatory behavior at the boundary and it is given as the region between two oscillatory functions, that is,

$$R^\epsilon = \{(x, y) \in \mathbb{R}^2 \mid x \in (0, 1), \ -\epsilon\, h_\epsilon(x) < y < \epsilon\, g_\epsilon(x)\}, \tag{3.0.2}$$

where $h_\epsilon(\cdot)$ and $g_\epsilon(\cdot)$ are functions satisfying $0 \le h_0 \le h_\epsilon(\cdot) \le h_1$, $0 < g_0 \le g_\epsilon(\cdot) \le g_1$ for some fixed constants h_0, h_1, g_0 and g_1, independent of $\epsilon > 0$, and such that oscillate as the parameter $\epsilon \to 0$.

Notice that the thickness of the domain has order ϵ, the upper boundary is defined by $\epsilon g_\epsilon(\cdot)$ and $-\epsilon h_\epsilon(\cdot)$ describes the lower boundary. We will allow g_ϵ and h_ϵ

to present different orders of frequency and profiles of oscillation, see Figure 3.1. We express this fact assuming that

$$g_\epsilon(x) = g\Big(\frac{x}{\epsilon^\beta}\Big), \qquad h_\epsilon(x) = h\Big(\frac{x}{\epsilon^\alpha}\Big),$$

where $\alpha, \beta > 0$ and the functions $g(\cdot)$ and $h(\cdot)$ satisfy the following hypothesis

(H) $g, h : \mathbb{R} \to \mathbb{R}$ are C^1 periodic functions with period L_1 and L_2 respectively such that there exist constants $h_0 \geq 0$ and $h_1, g_0, g_1 > 0$ verifying

$$0 \leq h_0 \leq h(\cdot) \leq h_1,$$

$$0 < g_0 \leq g(\cdot) \leq g_1,$$

where $h_0 = \min_{x\in\mathbb{R}}\{h(x)\}$ and $g_0 = \min_{x\in\mathbb{R}}\{g(x)\}$

Observe that the period of the oscillations at the top boundary has order ϵ^β while the period of the oscillations at the bottom boundary presents order ϵ^α where α and β are parameters greater than zero.

In this chapter we will treat the following cases:

(RF) Resonant-Fast oscillations: $\beta = 1$, $\alpha > 1$

(WF) Weak-Fast oscillations: $\beta < 1$, $\alpha > 1$

(FF) Fast-Fast oscillations: $\beta > 1$, $\alpha > 1$

(WW) Weak-Weak oscillations: $\beta < 1$, $\alpha < 1$.

The other two cases (Weak-resonant and resonant-resonant) are still under research.

Figure 3.1: Thin domain R^ϵ with doubly oscillatory boundary

When addressing these problems it is worth mentioning that we cannot apply a direct homogenization technique to obtain the limit problem since we do not have a basic cell. As a matter of fact we will need to combine different techniques. For instances, for the first case, (RF), we will use the classical oscillatory test functions method of Tartar together with an adaptation of the method from [10] to this new situation. For the case (WF) and (FF) we will use the unfolding method although not directly as in the case of Chapter 1. For the case (WW) we will transform the domain into the square $Q = (0, 1) \times (0, 1)$ and will be able to pass to the limit.

Moreover, notice that the effect of the oscillations at both boundaries, top and bottom, cannot be separated in an easy way since at the same time that the boundaries oscillate, the domain is shrinking and therefore the effect of the oscillations at one boundary is coupled in a nontrivial way with the oscillations at the other boundary. In this respect, this situation is quite different from the case where we have a fixed domain (say a rectangle $R_0 = (0,1)\times(0,b)$ and then the top and bottom boundary oscillates but without the domain shrinking in the vertical direction. In this case, the effect of the oscillating boundaries can be isolated. But in our case it is not possible. As a matter of fact, if we look at the limit problem, which is one dimensional of the form $-du_{xx} + u$, the diffusion coefficient d is always a constant and for the first three cases it has the following definition:

For the first case (RF), we have (see Section 3.1)

$$d = \frac{\hat{q}}{\mathcal{M}(g) + \mathcal{M}(h)}$$

where $\hat{q}$ is obtained via the solution of an appropriate elliptic problem in the "cell"

$$Y^* = \{(y_1, y_2) : 0 < y_1 < L_1, -h_0 < y_2 < g(y_1)\}$$

For the case (WF), we have (see Section 3.2)

$$d = \frac{1}{\mathcal{M}(\frac{1}{g+h_0})\mathcal{M}(g) + \mathcal{M}(h)}$$

For the case (FF), we have (see Section 3.3)

$$d = \frac{g_0 + h_0}{\mathcal{M}(g) + \mathcal{M}(h)}$$

Moreover, for the case (WW) and in the nontrivial case where $\alpha = \beta < 1$ but L_1 and L_2 are rationally independent, we have (see Section 3.4)

$$d = \frac{p_0}{\mathcal{M}(g) + \mathcal{M}(h)}$$

where

$$\frac{1}{p_0} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{g(y) + h(y)}dy$$

It is clear that oscillations from both boundaries are present in the limit problem through the diffusion coefficient and they are not trivially coupled.

We describe now the contents of the chapter.

- In Section 3.1 we address the case where the thin domain presents fast and resonant oscillations. Thus, the amplitude and period of the oscillations at the upper boundary, given by $\epsilon g(x/\epsilon)$, have also order ϵ while for the lower boundary, which is given by $-\epsilon h\big(x/\epsilon^\alpha\big)$, the period has order ϵ^α, with $\alpha > 1$.

- In Subsection 3.1.1 we fix the notation and state our main result on the convergence of the solutions, see Theorem 3.1.5. We also show some technical results which will be used in the proof of Theorem 3.1.5.
- Subsection 3.1.2 focuses on the proof of Theorem 3.1.5.
- Subsection 3.1.3 is devoted to a corrector result, we get strong convergence in H^1-norm for the solutions of problem (3.0.1) when we add the suitable corrector function. Let us point out that the construction of the corrector function is not standard and we believe that it is an important contribution of this chapter. In fact, the corrector approach developed in this chapter allows us also to obtain the first order corrector for the particular case where the thin domain presents only an extremely high oscillatory boundary which was not known in the literature.
- Subsection 3.1.4 generalizes the results of previous subsections to certain perforated domains.

Note that the contents of the first two subsections appear in [12].

- In Section 3.2 we combine the unfolding method and the techniques developed in the previous section to obtain the homogenized limit problem corresponding to (3.0.1) as the thin domain presents an extremely high oscillatory behavior at the bottom boundary and weak oscillations at the top boundary. More precisely, the period of the oscillations at the upper boundary has order ϵ^β with $0 < \beta < 1$ while the period at the lower boundary has order ϵ^α with $\alpha > 1$. The results of this section appears in [14].
- In Section 3.3 we complete the study started in two previous sections on the homogenization of thin domains with fast oscillations at the upper and lower boundary. We use the unfolding operator method to obtain the homogenized limit problem for the case where the top and bottom boundary present order of period ϵ^β and ϵ^α with $\alpha, \beta > 1$.
- Finally, in Section 3.4 we analyze the case of weak oscillations at the top and bottom boundary, namely the period of the oscillations at the upper and the lower boundary have order ϵ^β and ϵ^α with $0 < \alpha, \beta < 1$. Note that the method introduced in this section are based in a rescaling method as can be seen in classical works in thin domains [73, 100, 7]. This technique is very different to the techniques used in previous sections. Moreover, we will distinguish two cases according to $\alpha \neq \beta$ or $\alpha = \beta$. In this last situation we contemplate the possibility that both boundaries oscillate with different rationally independent periods, which amounts to study a quasi-periodic situation.

3.1. Fast and resonant oscillations

In this section we are interested in analyzing the behavior of solutions of problem (3.0.1) as the thin domain R^ϵ given by (3.0.2) presents at the lower boundary an extremely high oscillatory behavior and at the upper boundary oscillations with the same order of frequency as the thickness of the domain.

Then, using the notation above, the upper boundary is defined by the function g_ϵ which will present oscillations with frequency of the order of ϵ, that is,

$$g_\epsilon(x) = g\Big(\frac{x}{\epsilon}\Big),$$

where $g \in C^1(\mathbb{R})$ is a L_1-periodic function.

On the other hand, the lower boundary which is given by h_ϵ will present oscillations whose order of frequency is larger than the order of the compression of the thin domain, that is,

$$h_\epsilon(x) = h\Big(\frac{x}{\epsilon^\alpha}\Big),$$

where $\alpha > 1$, $h \in C^1(\mathbb{R})$ is a L_2-periodic function.

Since the domain R^ϵ is thin, it is reasonable to expect that the family of solutions of (3.0.1) will converge to a function of just one variable and that this function will satisfy certain elliptic equation in one dimension with some boundary conditions.

On one hand, if the domain does not present oscillations in the lower boundary, for instance consider the case $h_\epsilon(\cdot) \equiv 0$, the domain is periodic and there is a representative cell which describes the domain

$$Y^* = \{(y_1, y_2) \in \mathbb{R}^2 \mid 0 < y_1 < L_1,\ 0 < y_2 < g(y_1)\}.$$

In this case the limit equation is given by

$$\begin{cases} -q_0 w_{xx} + w = \dfrac{L_1}{|Y^*|}\hat{f}(x), \quad x \in (0,1) \\ w'(0) = w'(1) = 0 \end{cases} \tag{3.1.1}$$

where

$$q_0 = \frac{1}{|Y^*|}\int_{Y^*}\Big\{1 - \frac{\partial X}{\partial y_1}(y_1, y_2)\Big\}dy_1 dy_2,$$

the function $\hat{f}(x)$ is such that $\hat{f}^\epsilon \stackrel{\epsilon\to 0}{\rightharpoonup} \hat{f}$, w-$L^2(0,1)$, with $\hat{f}^\epsilon(x) = \int_0^{g(x/\epsilon)} f^\epsilon(x,y)\,dy$ and X is the unique solution L_1-periodic in the first variable of the following problem

$$\begin{cases} -\Delta X = 0 \text{ in } Y^*, \\ \dfrac{\partial X}{\partial N} = 0 \text{ on } B_2, \\ \dfrac{\partial X}{\partial N} = N_1 \text{ on } B_1, \\ \displaystyle\int_{Y^*} X\ dy_1 dy_2 = 0, \end{cases} \tag{3.1.2}$$

where B_1 and B_2 are the upper and the lower boundary of Y^* respectively. For details, we refer to [8, 84] and to Section 1.2 of Chapter 1 where we recover this well-known homogenized limit problem using the unfolding method, even for more general functions $g(\cdot)$.

On the other hand, the case where upper boundary is not oscillatory, so that the thin domain is given by

$$R^\epsilon = \{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ -\epsilon\, h\big(x/\epsilon^\alpha\big) < y < \epsilon\, G(x)\},$$

for some smooth function $G(\cdot)$ was treated in [10]. If $h_0 = \min_{x\in\mathbb{R}}\{h(x)\}$ then the variational formulation of the limit problem is:

$$\int_0^1 \Big\{ \Big(G(x) + h_0 \Big)\, w_x(x)\, \varphi_x(x) + p(x)\, \omega(x)\, \varphi(x) \Big\} dx = \int_0^1 \hat{f}(x)\, \varphi\, dx,\ \forall \varphi \in H^1(0,1), \tag{3.1.3}$$

where

$$p(x) = G(x) + \frac{1}{L_2} \int_0^{L_2} h(s)\, ds, \text{ for all } x \in (0,1),$$

and the function $\hat{f}(x)$ is such that $\hat{f}^\epsilon \overset{\epsilon \to 0}{\rightharpoonup} \hat{f}$, w-$L^2(0,1)$, with

$$\hat{f}^\epsilon(x) = \int_{-h\left(x/\epsilon^\alpha\right)}^{G(x)} f^\epsilon(x,y)\, dy,$$

(see [10] for details and for a more general result).

As a matter of fact, this case is a combination of the two cases described above and we will need to employ a combination of the techniques used in the proof of the two cases separately to obtain the homogenized limit problem. Actually, one of the key points to obtain the homogenization result is the construction of the oscillating test functions which allows us to pass to the limit. Notice that, to define the appropriate test functions we will use in an essential way that the thin domain presents much more oscillations at the bottom than at the top boundary.

In addition, it is worth observing that the construction of the appropriate oscillating test functions will allow us to address the question of correctors in Subsection 3.1.3.

Finally we would like to point that the techniques introduced in the first three subsections can be applied to other related problems. For instance, in Subsection 3.1.4 we show how to adapt the method for certain perforated thin domains and in [95] our approach is used to study the locally periodic case where the amplitude of the oscillations at the top and the bottom boundary depends on x. Actually, the author uses our approach to solve first the piecewise periodic case and then, an approximation argument introduced in [9] is used to get the limit problem for this locally periodic case.

3.1.1. Notation, important facts and statement of the main result

In this section we set up the problem, describing clearly the domain and the equations we are dealing with. Furthermore, we state the main convergence result, Theorem 3.1.5, and some technical results needed for its proof in the next section.

First of all, we start describing in detail the thin domain. As we have mentioned, we consider two positive functions $g, h : \mathbb{R} \to \mathbb{R}$ with period L_1 and L_2 respectively such that they belong to $C^1(\mathbb{R})$ and there exist constants $h_0 \geq 0$ and $h_1, g_0, g_1 > 0$ satisfying

$$0 \leq h_0 \leq h(\cdot) \leq h_1,$$

$$0 < g_0 \leq g(\cdot) \leq g_1,$$

where $h_0 = \min_{x\in\mathbb{R}}\{h(x)\}$ and $g_0 = \min_{x\in\mathbb{R}}\{g(x)\}$. Then, we define the thin domain as follows

$$R^\epsilon = \Big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ -\epsilon\, h\Big(\frac{x}{\epsilon^\alpha}\Big) < y < \epsilon\, g\Big(\frac{x}{\epsilon}\Big)\Big\}, \quad \text{with } \alpha > 1. \tag{3.1.4}$$

Remark 3.1.1. *Notice that by the Average Convergence for Periodic Functions Theorem (see, e.g., [52, p. xvi]) we have*

$$g\Big(\frac{\cdot}{\epsilon}\Big) \overset{\epsilon\to 0}{\rightharpoonup} \frac{1}{L_1}\int_0^{L_1} g(s)\, ds \quad w^* - L^\infty(I), \tag{3.1.5}$$

$$h\Big(\frac{\cdot}{\epsilon^\alpha}\Big) \overset{\epsilon\to 0}{\rightharpoonup} \frac{1}{L_2}\int_0^{L_2} h(s)\, ds \quad w^* - L^\infty(I). \tag{3.1.6}$$

To study the convergence of the solutions of (3.0.1) we first perform the change of variables $(x,y) \to (x, y/\epsilon)$, which transforms the domain R^ϵ into the domain Ω^ϵ, see Figure 3.2, given by

$$\Omega^\epsilon = \Big\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ -h\Big(\frac{x_1}{\epsilon^\alpha}\Big) < x_2 < g\Big(\frac{x_1}{\epsilon}\Big)\Big\}. \tag{3.1.7}$$

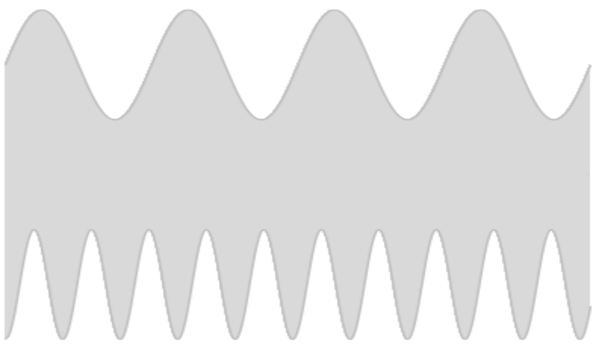


Figure 3.2: Domain Ω^ϵ with resonant and fast oscillations

Under this transformation, we obtain the equivalent linear elliptic problem

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f^\epsilon & \text{in } \Omega^\epsilon, \\ \dfrac{\partial u^\epsilon}{\partial x_1}\mu_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2}\mu_2^\epsilon = 0 & \text{on } \partial\Omega^\epsilon, \end{cases} \tag{3.1.8}$$

where $f^\epsilon \in L^2(\Omega^\epsilon)$ satisfies $\|f^\epsilon\|_{L^2(\Omega^\epsilon)} \leq C$, for some $C > 0$ independent of ϵ, and $\mu^\epsilon = (\mu_1^\epsilon, \mu_2^\epsilon)$ is the outward unit normal to $\partial\Omega^\epsilon$.

Observe that by changing the scale of the domain R^ϵ we obtain the domain Ω^ϵ which is not a thin domain anymore although, to balance this, there appears a factor $1/\epsilon^2$ in front of the second derivative with respect to the variable x_2, which means a very fast diffusion in the vertical direction. Moreover, Ω^ϵ has very wild oscillatory behavior at the upper and lower boundary.

For the analysis below, we will construct an extension operator, P_ϵ, which will extend functions defined in the domain Ω^ϵ to functions defined in the domain

$$\widetilde{\Omega}^\epsilon = \Big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ -h\big(x_1/\epsilon^\alpha\big) < x_2 < g_1\Big\}. \tag{3.1.9}$$

Lemma 3.1.2. *With the notation above, there exists an extension operator*

$$P_\epsilon \in \mathcal{L}(L^p(\Omega^\epsilon), L^p(\widetilde{\Omega}^\epsilon)) \cap \mathcal{L}(W^{1,p}(\Omega^\epsilon), W^{1,p}(\widetilde{\Omega}^\epsilon))$$

such that for any $\varphi \in W^{1,p}(\Omega^\epsilon)$,

$$\begin{aligned}
&||P_\epsilon\varphi||_{L^p(\widetilde{\Omega}^\epsilon)} \le C||\varphi||_{L^p(\Omega^\epsilon)},\\
&\left\|\frac{\partial P_\epsilon\varphi}{\partial x_1}\right\|_{L^p(\widetilde{\Omega}^\epsilon)} \le C\left\{\left\|\frac{\partial\varphi}{\partial x_1}\right\|_{L^p(\Omega^\epsilon)} + \frac{1}{\epsilon}\left\|\frac{\partial\varphi}{\partial x_2}\right\|_{L^p(\Omega^\epsilon)}\right\},\\
&\left\|\frac{\partial P_\epsilon\varphi}{\partial x_2}\right\|_{L^p(\widetilde{\Omega}^\epsilon)} \le C\left\|\frac{\partial\varphi}{\partial x_2}\right\|_{L^p(\Omega^\epsilon)},
\end{aligned} \tag{3.1.10}$$

where $1 \le p \le \infty$ *and* C *is a constant independent of* ϵ.

Proof. The construction of the extension operator P_ϵ is essentially based on a reflection technique in the x_2 direction along the upper oscillating boundary, as in [8, 9].

Let us consider two cases. On one hand, if we have that $g_1 - h_0 \le 2g_0$, which implies that $g(x_1/\epsilon) - h(x_1/\epsilon^\alpha) \le 2g_0$, we can define the extension operator as follows

$$(P_\epsilon\varphi)(x_1, x_2) = \begin{cases} \varphi(x_1, x_2) & (x_1, x_2) \in \Omega^\epsilon \\ \varphi\big(x_1, 2g(x_1/\epsilon) - x_2\big) & (x_1, x_2) \in \widetilde{\Omega}^\epsilon \setminus \Omega^\epsilon, \end{cases}$$

for any $\varphi \in W^{1,p}(\Omega^\epsilon)$. It easily follows that this operator satisfies the inequalities (3.1.10).

On the other hand, if we are in the case where $g_1 - h_0 > 2g_0$, we will need to extend first the function φ in the direction of negative x_2. Hence, let us consider the following subset of Ω^ϵ

$$\Omega_0^\epsilon = \Big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ -h_0 < x_2 < g\Big(\frac{x_1}{\epsilon}\Big)\Big\}. \tag{3.1.11}$$

We extend $\varphi_{|\Omega_0^\epsilon}$ to the set

$$\Big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ -h_0 - g_0 < x_2 < g\Big(\frac{x_1}{\epsilon}\Big)\Big\}$$

with the reflecting along the line $x_2 = -h_0$. This produces the function

$$\varphi_1(x_1, x_2) = \begin{cases} \varphi(x_1, x_2) & \text{if } (x_1, x_2) \in \Omega_0^\epsilon, \\ \varphi(x_1, -x_2 - 2h_0) & \text{if } -h_0 - g_0 < x_2 < -h_0. \end{cases}$$

We can continue producing these reflections inductively as follows

$$\varphi_n(x_1, x_2) = \begin{cases} \varphi_{n-1}(x_1, x_2) & \text{if } -h_0 - (n-1)g_0 < x_2 < g(x_1/\epsilon), \\ \varphi_{n-1}\big(x_1, -x_2 - 2(n-1)g_0 - 2h_0\big) & \text{if } -h_0 - ng_0 < x_2 < -h_0 - (n-1)g_0. \end{cases}$$

Choosing n large enough so that $ng_0 > g_1 - h_0$, constructing φ_n and applying to φ_n the procedure by reflection in the x_2 direction along the oscillating upper boundary, we obtain the extension operator which satisfies the inequalities (3.1.10)

$$(P_\epsilon \varphi)(x_1, x_2) = \begin{cases} \varphi(x_1, x_2) & (x_1, x_2) \in \Omega^\epsilon \\ \varphi_n\big(x_1, 2g(x_1/\epsilon) - x_2\big) & (x_1, x_2) \in \widetilde{\Omega}^\epsilon \setminus \Omega^\epsilon. \end{cases}$$

□

Remark 3.1.3. *Observe that the extension operator P_ϵ preserves periodicity in the x_1 variable. Indeed, if the function φ is periodic in x_1, then the extended function $P_\epsilon \varphi$ is periodic in x_1 too.*

Furthermore, note that this procedure can also be applied to the case where the functions which define the oscillatory boundaries are independent of ϵ, $h_\epsilon(x) = h(x)$ and $g_\epsilon(x) = g(x)$. For example, the same construction may be used for the representative cell $Y^* = \{(y_1, y_2) \in \mathbb{R}^2 \mid 0 < y_1 < L_1,\ -h_0 < y_2 < g(y_1)\}$. Then, using the same reflection procedure as in Lemma 3.1.2 we may define an extension operator $P \in \mathcal{L}(H^1(Y^*), H^1(Y)) \cap \mathcal{L}(L^2(Y^*), L^2(Y))$, such that for any $\varphi \in H^1(Y^*)$

$$\begin{aligned} &||P\varphi||_{L^p(Y)} \le C||\varphi||_{L^p(Y^*)}, \\ &\|\nabla(P\varphi)\|_{[L^p(Y)]^2} \le C\|\nabla\varphi\|_{[L^p(Y^*)]^2}, \end{aligned}$$

where $Y = (0, L_1) \times (-h_0, g_1)$.

Before we state the main result we introduce a technical result originally from [10] which we will use later in the proofs. Let us show some relevant estimates on the solutions of certain elliptic problems posed in rectangles of the type

$$Q_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid -\epsilon^\alpha < x < \epsilon^\alpha,\ 0 < y < 1\}, \text{ with } \alpha > 1. \tag{3.1.12}$$

As a matter of fact, for $w_0(\cdot) \in H^1(-\epsilon^\alpha, \epsilon^\alpha)$, we define the function $w^\epsilon(x, y)$ as the unique solution of

$$\begin{cases} \dfrac{\partial^2 w^\epsilon}{\partial x^2} + \dfrac{1}{\epsilon^2}\dfrac{\partial^2 w^\epsilon}{\partial y^2} = 0 & \text{in } Q_\epsilon, \\ w^\epsilon(x, 0) = w_0(x) & \text{on } \Gamma_\epsilon, \\ \dfrac{\partial w^\epsilon}{\partial \nu} = 0 & \text{on } \partial Q_\epsilon \setminus \Gamma_\epsilon, \end{cases} \tag{3.1.13}$$

where ν is the outward unit normal to ∂Q_ϵ and $\Gamma_\epsilon = \{(x, 0) \in \mathbb{R}^2 \mid -\epsilon^\alpha < x < \epsilon^\alpha\}$.

We have the following

Lemma 3.1.4. *With the notation from above, if we denote by $\bar{w}_0$ the average of w_0 in Γ_ϵ, that is*

$$\bar{w}_0 = \frac{1}{2\epsilon^\alpha}\int_{-\epsilon^\alpha}^{\epsilon^\alpha} w_0(x)\,dx,$$

then there exists a constant C, independent of ϵ and w_0, such that

$$\int_0^1\int_{-\epsilon^\alpha}^{\epsilon^\alpha} |w^\epsilon(x,y)-\bar{w}_0|^2\,dxdy \le C\epsilon^{\alpha-1}\|w_0\|^2_{L^2(-\epsilon^\alpha,\epsilon^\alpha)} \tag{3.1.14}$$

and

$$\left\|\frac{\partial w^\epsilon}{\partial x}\right\|^2_{L^2(Q_\epsilon)} + \frac{1}{\epsilon^2}\left\|\frac{\partial w^\epsilon}{\partial y}\right\|^2_{L^2(Q_\epsilon)} \le C\epsilon^{\alpha-1}\left\|\frac{\partial w_0}{\partial x}\right\|^2_{L^2(-\epsilon^\alpha,\epsilon^\alpha)}. \tag{3.1.15}$$

Proof. See [10, Lemma 3.1]. The proof of this result is based in the known fact that the solution of the problem (3.1.13) can be found explicitly and admits a Fourier decomposition of the form

$$w^\epsilon(x,y) = \frac{1}{2\epsilon^\alpha}\int_{-\epsilon^\alpha}^{\epsilon^\alpha} w_0(x)dx + \sum_{k=1}^{\infty}(w_0,\varphi_n^\epsilon)\varphi_n^\epsilon(x)\frac{\cosh(\frac{n\pi(1-y)}{\epsilon^{\alpha-1}})}{\cosh(\frac{n\pi}{\epsilon^{\alpha-1}})}$$

where $\varphi_n^\epsilon(x) = \epsilon^{-\alpha/2}\cos(\frac{n\pi x}{\epsilon^\alpha})$, $n = 1,2,\dots$, and $(w_0,\varphi_n^\epsilon) = (w_0,\varphi_n^\epsilon)_{L^2(-\epsilon^\alpha,\epsilon^\alpha)}$ □

We now introduce the main result of this section.

Theorem 3.1.5. *Let u^ϵ be the unique solution of* (3.1.8)*. Assume that $f^\epsilon \in L^2(\Omega^\epsilon)$ satisfies $\|f^\epsilon\|_{L^2(\Omega^\epsilon)} \le C$ with C independent of the parameter ϵ, and there exists $\hat{f} \in L^2(0,1)$ such that $\hat{f}^\epsilon \overset{\epsilon\to 0}{\rightharpoonup} \hat{f}$ $w-L^2(0,1)$, where*

$$\hat{f}^\epsilon(x_1) \equiv \int_{-h(x_1/\epsilon^\alpha)}^{g(x_1/\epsilon)} f^\epsilon(x_1,x_2)\,dx_2.$$

Then, there exists $u_0 \in H^1(0,1)$ such that, if P_ϵ is the extension operator constructed in Lemma 3.1.2 we have

$$\|P_\epsilon u^\epsilon - u_0\|_{L^2(\widetilde{\Omega}^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0. \tag{3.1.16}$$

Furthermore, $u_0 \in H^1(0,1)$ is the unique weak solution of the following Neumann problem

$$\begin{cases} -\dfrac{\hat{q}}{\frac{|Y^*|}{L_1}+p}u_{0xx}(x) + u_0(x) = \dfrac{\hat{f}(x)}{\frac{|Y^*|}{L_1}+p}, \quad x\in(0,1), \\ u_0'(0) = u_0'(1) = 0, \end{cases} \tag{3.1.17}$$

where the homogenized constant coefficients are defined by

$$\begin{aligned} \hat{q} &= \frac{1}{L_1}\int_{Y^*}\Big\{1 - \frac{\partial X}{\partial y_1}(y_1,y_2)\Big\}\,dy_1dy_2, \\ p &= \frac{1}{L_2}\int_0^{L_2} h(s)\,ds - h_0, \end{aligned} \tag{3.1.18}$$

Y^* is the representative cell given by

$$Y^* = \{(y_1, y_2) \in \mathbb{R}^2 \mid 0 < y_1 < L_1,\ -h_0 < y_2 < g(y_1)\}, \tag{3.1.19}$$

and X is the unique solution (up to constants) which is L_1-periodic in the first variable, of the problem

$$\begin{cases} -\Delta X = 0 \ \text{in } Y^*, \\ \dfrac{\partial X}{\partial N} = 0 \ \text{on } B_2, \\ \dfrac{\partial X}{\partial N} = -\dfrac{g'(y_1)}{\sqrt{1+g'(y_1)^2}} \ \text{on } B_1, \end{cases} \tag{3.1.20}$$

where B_1 is the upper boundary and B_2 is the lower boundary of ∂Y^*.

Remark 3.1.6. *Observe that the diffusion coefficient reflects the doubly oscillatory behavior of the domain. One can clearly distinguish two kind of terms: $\hat{q}$ from the purely periodic oscillatory boundary, upper boundary, and p from the highly oscillatory boundary, lower boundary.*

Remark 3.1.7. *If the non homogeneous term $f^\epsilon(x_1, x_2)$ is a fixed function depending only on the first variable, that is, $f^\epsilon(x_1, x_2) = f(x_1)$, it is easy to see that*

$$\hat{f}(x_1) = \left(\frac{|Y^*|}{L_1} + p\right) f(x_1)$$

and therefore, (3.1.17) *can be written as*

$$\begin{cases} -\dfrac{\hat{q}}{\frac{|Y^*|}{L_1} + p} u_{0xx} + u_0 = f(x) \quad x \in (0,1), \\ u_0'(0) = u_0'(1) = 0. \end{cases} \tag{3.1.21}$$

Remark 3.1.8. *Notice that p in* (3.1.18) *can be written as $p = \frac{1}{L_2}\int_0^{L_2}(h(s) - h_0)$. Therefore, if we define the "fast oscillating cell" from the bottom boundary as*

$$Y_-^* = \{(y_1, y_2) : 0 < y_1 < L_2,\ -h(y_1) < y_2 < -h_0\}$$

then $p = \frac{|Y_-^*|}{L_2}$. Therefore, the limit homogenized equation (3.1.17) *can also be written as*

$$\begin{cases} -\dfrac{\hat{q}}{\frac{|Y^*|}{L_1} + \frac{|Y_-^*|}{L_2}} u_{0xx}(x) + u_0(x) = \dfrac{\hat{f}(x)}{\frac{|Y^*|}{L_1} + \frac{|Y_-^*|}{L_2}}, \quad x \in (0,1), \\ u_0'(0) = u_0'(1) = 0, \end{cases} \tag{3.1.22}$$

which express more clearly the geometry of the thin domain.

Also, it may be interesting to note that

$$\frac{|Y^*|}{L_1} + \frac{|Y_-^*|}{L_2} = \mathcal{M}(g) + \mathcal{M}(h)$$

where the operator $\mathcal{M}$ applied to a periodic function, gives us the average of the function on a period (see the Notation Section).

Remark 3.1.9. *Notice that in case* $h(\cdot) \equiv 0$ *we get* $p = 0$ *and we recover the homogenized limit problem* (3.1.1)*, see Section 1.2 of Chapter 1. Furthermore, if the upper boundary does not oscillate, say the function* g *is the constant function* $g(x) \equiv g_0 > 0$*, then the function* X *above is constant and therefore* $\hat{q} = \frac{|Y^*|}{L_1} = g_0 + h_0$*. Hence, in this case the equation* (3.1.21) *is*

$$\begin{cases} -\dfrac{g_0 + h_0}{g_0 + \int_0^1 h(s)ds} u_{0xx} + u_0 = f(x) \quad x \in (0,1), \\ \qquad\qquad u_0'(0) = u_0'(1) = 0, \end{cases}$$

which coincides with the equation obtained in [10, Corollary 2.3] for thin domains only with a very high oscillatory boundary. See also Remark 1.4.4 in Chapter 1 where the same homogenized limit problem is obtained using the unfolding operator method.

Finally, we would like to point out that we may consider different conditions in the lateral boundaries of the thin domain R^ϵ while we preserve the Neumann boundary condition in the upper and lower boundary. In fact, the limit problem preserves the boundary condition, for instance, if we consider conditions of Dirichlet type, $u^\epsilon = 0$ or even Robin, $\frac{\partial u^\epsilon}{\partial N^\epsilon} + \gamma u^\epsilon = 0$, in the lateral boundary of the problem (3.1.8), then, the limit problem will preserve this boundary condition. Moreover, for the case with homogeneous Dirichlet condition on the top and bottom is obvious that $u_0 = 0$ since the extension by zero of the solution now belongs to $H_0^1\big((0,1) \times (-h_1, g_1)\big)$.

3.1.2. Proof of the main result

In this subsection we prove in detail Theorem 3.1.5. As we have already mentioned we properly combine two different techniques to be able to get the homogenized limit equation of problem (3.1.8). On one hand, we use an extension operator in the upper boundary and the variational method of oscillating test functions due to Tartar. On the other hand, in order to study the behavior of the solutions close to the extremely high oscillatory boundary we define suitable rectangles containing the lower boundary to apply Lemma 3.1.4.

Proof of Theorem 3.1.5. The variational formulation of (3.1.8) is: find $u^\epsilon \in H^1(\Omega^\epsilon)$ such that

$$\int_{\Omega^\epsilon} \Big\{ \frac{\partial u^\epsilon}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2} \frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi \Big\} dx_1 dx_2 = \int_{\Omega^\epsilon} f^\epsilon \varphi dx_1 dx_2, \quad \forall \varphi \in H^1(\Omega^\epsilon). \tag{3.1.23}$$

Taking $\varphi = u^\epsilon$ in expression (3.1.23) and using that $\|f^\epsilon\|_{L^2(\Omega^\epsilon)} \leq C$, we easily obtain the a priori bounds

$$\|u^\epsilon\|_{L^2(\Omega^\epsilon)}, \quad \Big\|\frac{\partial u^\epsilon}{\partial x_1}\Big\|_{L^2(\Omega^\epsilon)}, \quad \frac{1}{\epsilon}\Big\|\frac{\partial u^\epsilon}{\partial x_2}\Big\|_{L^2(\Omega^\epsilon)} \leq C. \tag{3.1.24}$$

If we denote by $\tilde{}$ the standard extension by zero, by χ^ϵ the characteristic function of Ω^ϵ and we use the extension operator P_ϵ constructed in Lemma 3.1.2, we may write (3.1.23) as

$$\int_{\Omega_0} \left\{ \widetilde{\frac{\partial u^\epsilon}{\partial x_1}} \frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2} \widetilde{\frac{\partial u^\epsilon}{\partial x_2}} \frac{\partial \varphi}{\partial x_2} \right\} dx_1 dx_2 + \int_{\widetilde{\Omega}^\epsilon_-} \left\{ \frac{\partial u^\epsilon}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2} \frac{\partial \varphi}{\partial x_2} \right\} dx_1 dx_2$$
$$+ \int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon P_\epsilon u^\epsilon \varphi \, dx_1 dx_2 = \int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon f^\epsilon \varphi \, dx_1 dx_2, \quad \forall \varphi \in H^1(\Omega^\epsilon), \tag{3.1.25}$$

where we divide the domain $\widetilde{\Omega}^\epsilon$ (see (3.1.9)) in two parts, see Figure 3.3: one of them, $\widetilde{\Omega}^\epsilon_-$, presents oscillations and the other one, Ω_0, is a fixed domain, that is,

$$\widetilde{\Omega}^\epsilon_- = \left\{ (x_1, x_2) \in \mathbb{R}^2 | \, x_1 \in (0,1), \; -h\left(\frac{x_1}{\epsilon^\alpha}\right) < x_2 < -h_0 \right\},$$
$$\Omega_0 = \{ (x_1, x_2) \in \mathbb{R}^2 | \, x_1 \in (0,1), \; -h_0 < x_2 < g_1 \}.$$

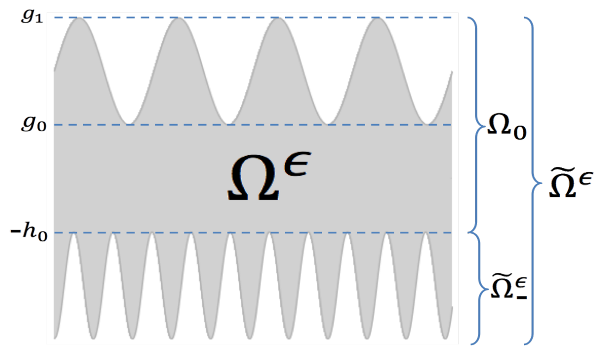


Figure 3.3: Sets $\Omega^\epsilon, \widetilde{\Omega}^\epsilon, \widetilde{\Omega}^\epsilon_-$ and Ω_0

Now, we study the limit of the different functions that form the integrands of (3.1.25).

(a) Limit in the extended functions.

Using the a priori estimate (3.1.24) and the results from Lemma 3.1.2 on the extension operator P_ϵ we get that

$$\|P_\epsilon u^\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)}, \left\| \frac{\partial P_\epsilon u^\epsilon}{\partial x_1} \right\|_{L^2(\widetilde{\Omega}^\epsilon)} \text{ and } \frac{1}{\epsilon} \left\| \frac{\partial P_\epsilon u^\epsilon}{\partial x_2} \right\|_{L^2(\widetilde{\Omega}^\epsilon)} \le C \quad \text{for all } \epsilon > 0, \tag{3.1.26}$$

where C is a positive constant independent of ϵ.

Consequently, it follows that $P_\epsilon u^\epsilon|_{\Omega_0} \in H^1(\Omega_0)$ satisfies for all $\epsilon > 0$

$$\|P_\epsilon u^\epsilon\|_{L^2(\Omega_0)}, \left\| \frac{\partial P_\epsilon u^\epsilon}{\partial x_1} \right\|_{L^2(\Omega_0)} \text{ and } \frac{1}{\epsilon} \left\| \frac{\partial P_\epsilon u^\epsilon}{\partial x_2} \right\|_{L^2(\Omega_0)} \le C.$$

Therefore, we can extract a subsequence of $\{P_\epsilon u^\epsilon|_{\Omega_0}\} \subset H^1(\Omega_0)$, denoted again by

$\{P_\epsilon u^\epsilon\}$, such that

$$\begin{aligned}
&P_\epsilon u^\epsilon \overset{\epsilon\to 0}{\rightharpoonup} u_0 \quad w - H^1(\Omega_0),\\
&P_\epsilon u^\epsilon \overset{\epsilon\to 0}{\longrightarrow} u_0 \quad s - H^s(\Omega_0) \text{ for all } s \in [0,1) \text{ and}\\
&\frac{\partial P_\epsilon u^\epsilon}{\partial x_2} \overset{\epsilon\to 0}{\longrightarrow} 0 \quad s - L^2(\Omega_0),
\end{aligned} \tag{3.1.27}$$

for some $u_0 \in H^1(\Omega_0)$.

As a consequence of (3.1.27), we have that the function $u_0(x_1, x_2)$ does not depend on the variable x_2, that is,

$$\frac{\partial u_0}{x_2}(x_1, x_2) = 0 \text{ a.e. } \Omega_0,$$

and then, $u_0(x_1, x_2) = u_0(x_1)$ for a.e. $(x_1, x_2) \in \Omega_0$.

Moreover, from (3.1.27) and using the properties of the trace, we have that the restriction of $P_\epsilon u^\epsilon$ to $\Gamma = \{(x_1, 0) \in \mathbb{R}^2 \,|\, x_1 \in (0,1)\}$ converges to u_0, that is,

$$P_\epsilon u^\epsilon|_\Gamma \overset{\epsilon\to 0}{\longrightarrow} u_0 \quad s - H^s(\Gamma)$$

for all $s \in [0, 1/2)$. Hence, we get

$$\|P_\epsilon u^\epsilon - u_0\|_{L^2(\Gamma)} \overset{\epsilon\to 0}{\longrightarrow} 0.$$

In view of the limit above we show now that

$$\|P_\epsilon u^\epsilon - u_0\|_{L^2(\widetilde{\Omega}^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0. \tag{3.1.28}$$

Indeed, we have

$$\begin{aligned}
\|P_\epsilon u^\epsilon_{|\Gamma} - u_0\|^2_{L^2(\widetilde{\Omega}^\epsilon)} &= \int_0^1 \int_{-h(x_1/\epsilon^\alpha)}^{g_1} |P_\epsilon u^\epsilon(x_1, 0) - u_0(x_1)|^2 \, dx_2 dx_1\\
&\le (g_1 + h_1) \, \|P_\epsilon u^\epsilon - u_0\|^2_{L^2(\Gamma)} \overset{\epsilon\to 0}{\longrightarrow} 0.
\end{aligned}$$

Furthermore, by Hölder's inequality we obtain

$$\begin{aligned}
|P_\epsilon u^\epsilon(x_1, x_2) - P_\epsilon u^\epsilon(x_1, 0)|^2 &= \Big| \int_0^{x_2} \frac{\partial P_\epsilon u^\epsilon}{\partial x_2}(x_1, s) \, ds \Big|^2\\
&\le \left(\int_0^{x_2} \left| \frac{\partial P_\epsilon u^\epsilon}{\partial x_2}(x_1, s) \right|^2 ds \right) |x_2|.
\end{aligned}$$

Then, integrating in $\widetilde{\Omega}^\epsilon$ and using (3.1.26) we have

$$\begin{aligned}
\|P_\epsilon u^\epsilon - P_\epsilon u^\epsilon_{|\Gamma}\|^2_{L^2(\widetilde{\Omega}^\epsilon)} &= \int_0^1 \int_{-h(x_1/\epsilon^\alpha)}^{g_1} |P_\epsilon u^\epsilon(x_1, x_2) - P_\epsilon u^\epsilon(x_1, 0)|^2 \, dx_1 dx_2\\
&\le \int_0^1 \int_{-h(x_1/\epsilon^\alpha)}^{g_1} \left(\int_0^{x_2} \left| \frac{\partial P_\epsilon u^\epsilon}{\partial x_2}(x_1, s) \right|^2 ds \right) |x_2| \, dx_2 dx_1
\end{aligned}$$

$$\leq (g_1 + h_1)^2 \left\| \frac{\partial P_\epsilon u^\epsilon}{\partial x_2} \right\|^2_{L^2(\widetilde{\Omega}^\epsilon)} \to 0 \text{ as } \epsilon \to 0.$$

Finally, by Minkowski's inequality we get

$$\|P_\epsilon u^\epsilon - u_0\|_{L^2(\widetilde{\Omega}^\epsilon)} \leq \|P_\epsilon u^\epsilon - P_\epsilon u^\epsilon_{|\Gamma}\|_{L^2(\widetilde{\Omega}^\epsilon)} + \|P_\epsilon u^\epsilon_{|\Gamma} - u_0\|_{L^2(\widetilde{\Omega}^\epsilon)} \stackrel{\epsilon \to 0}{\longrightarrow} 0.$$

which shows (3.1.28).

(b) Limit in the tilde functions.

From the a priori estimates (3.1.24) we know by weak compactness that there exists a function $\xi^* \in L^2(\Omega_0)$, such that, up to subsequences

$$\widetilde{\frac{\partial u^\epsilon}{\partial x_1}} \stackrel{\epsilon \to 0}{\rightharpoonup} \xi^* \quad w - L^2(\Omega_0) \quad \text{and} \quad \widetilde{\frac{\partial u^\epsilon}{\partial x_2}} \stackrel{\epsilon \to 0}{\longrightarrow} 0 \quad s - L^2(\Omega_0). \tag{3.1.29}$$

(c) Limit of χ^ϵ.

Let χ be the characteristic function of the representative cell Y^*. Recall that Y^* is given by

$$Y^* = \{(y_1, y_2) \in \mathbb{R}^2 \mid 0 < y_1 < L_1,\, -h_0 < y_2 < g(y_1)\},$$

and Y^* somehow ignores the fast oscillatory behavior of the bottom boundary.

We extend χ periodically on the variable $y_1 \in \mathbb{R}$ and denote this extension again by χ. Recalling that χ^ϵ is the characteristic function of Ω^ϵ. Clearly, by construction,

$$\chi^\epsilon(x_1, x_2) = \chi(x_1/\epsilon, x_2), \text{ for } (x_1, x_2) \in \Omega_0.$$

Consequently, by the Average Convergence for Periodic Functions Theorem (see, e.g., [52, p. xvi]) we obtain

$$\chi^\epsilon(\cdot, x_2) \stackrel{\epsilon \to 0}{\rightharpoonup} \theta(x_2) := \frac{1}{L_1}\int_0^{L_1} \chi(s, x_2) ds \quad w^* - L^\infty(0,1), \quad \forall x_2 \in (-h_0, g_1). \tag{3.1.30}$$

Therefore, we have

$$H^\epsilon(x_2) = \int_0^1 \varphi(x_1, x_2) \Big\{ \chi^\epsilon(x_1, x_2) - \theta(x_2) \Big\}\, dx_1 \stackrel{\epsilon \to 0}{\longrightarrow} 0,$$

for any $\varphi \in L^1(\Omega_0)$ and for almost everywhere $x_2 \in (-h_0, g_1)$.

Then, taking into account the following two assertions

$$\int_{\Omega_0} \varphi(x_1, x_2) \Big\{ \chi^\epsilon(x_1, x_2) - \theta(x_2) \Big\}\, dx_1 dx_2 = \int_{-h_0}^{g_1} H^\epsilon(x_2) dx_2,$$

$$|H^\epsilon(x_2)| \leq \int_0^1 |\varphi(x_1, x_2)| dx_1,$$

we get by Lebesgue's Dominated Convergence Theorem that

$$\chi^\epsilon \stackrel{\epsilon \to 0}{\rightharpoonup} \theta \quad w^* - L^\infty(\Omega_0), \tag{3.1.31}$$

where $\theta(x_2) := \dfrac{1}{L_1}\displaystyle\int_0^{L_1} \chi(s, x_2)ds$, for $x_2 \in (-h_0, g_1)$. Moreover, with the definition of θ, simple computations show that

$$\int_{\Omega_0} \theta(x_2)\phi(x_1)dx_1dx_2 = \frac{|Y^*|}{L_1}\int_0^1 \phi(x_1)dx_1, \qquad \text{for all } \phi \in L^1(0,1) \tag{3.1.32}$$

(d) Test functions.

In order to construct appropriate test functions that will allow us to pass to the limit in the variational formulation (3.1.25), we are going to need to define a partition of the unit interval $[0,1]$ which is related to the function $h\big(\,\cdot\,/\epsilon^\alpha\big)$ and which will allow us to analyze in detail the effect of the oscillations at the bottom boundary in the limit equation using Lemma 3.1.4.

Hence, let us denote by N_ϵ the largest integer such that $N_\epsilon L_2 \epsilon^\alpha < 1$, where L_2 is the period of the function $h(\cdot)$. Observe that $N_\epsilon \sim L_2^{-1}\epsilon^{-\alpha}$.

Observe that in any interval of length $L_2\epsilon^\alpha$, the function $h(\frac{\cdot}{\epsilon^\alpha})$ has exactly one period and therefore at some point of the interval the minimum of h, that is h_0, is attained. Hence,

$$h_0 = \min_{x\in[(n-1)L_2\epsilon^\alpha, nL_2\epsilon^\alpha]} h\left(\frac{x}{\epsilon^\alpha}\right), \quad n = 1, 2\ldots, N_\epsilon \tag{3.1.33}$$

and this minimum is attained at a point $\gamma_{n,\epsilon} \in ((n-1)L_2\epsilon^\alpha, nL_2\epsilon^\alpha]$. This point may not be unique but again using the periodicity of $h(\frac{\cdot}{\epsilon^\alpha})$, we can take $\gamma_{n,\epsilon} = z_0\epsilon^\alpha + (n-1)L_2\epsilon^\alpha$, $n = 1, 2, \ldots, N_\epsilon$, where z_0 is a point in $(0, L_2]$ where h attains the value h_0. Then, we have

$$h\Big(\frac{\gamma_{n,\epsilon}}{\epsilon^\alpha}\Big) = h_0.$$

By extension, let us denote by $\gamma_{0,\epsilon} = 0$ and $\gamma_{N_\epsilon+1,\epsilon} = 1$. Note that the set $\{\gamma_{0,\epsilon}, \gamma_{1,\epsilon}, ..., \gamma_{N_\epsilon+1,\epsilon}\}$ defines a partition for the unit interval $[0,1]$.

Notice that by definition we have that

$$\{(\gamma_{n,\epsilon}, x_2) \mid -h_1 < x_2 < -h_0\} \cap \widetilde{\Omega}^\epsilon = \emptyset,$$

for all $n = 1, 2, \ldots, N_\epsilon$.

We define now the test functions as follows. Let $\phi \in H^1(0,1)$, we consider $\varphi^\epsilon \in H^1(\widetilde{\Omega}^\epsilon)$ defined as

$$\varphi^\epsilon(x_1, x_2) = \begin{cases} W_n^\epsilon(x_1, x_2), & (x_1, x_2) \in \widetilde{\Omega}_-^\epsilon \cap Q_n^\epsilon, \quad n = 0, 1, \ldots, N^\epsilon, \\ \phi(x_1), & (x_1, x_2) \in \Omega_0, \end{cases}$$

where Q_n^ϵ is the rectangle (see Figure 3.4)

$$Q_n^\epsilon = \{(x_1, x_2) \,|\, \gamma_{n,\epsilon} < x_1 < \gamma_{n+1,\epsilon}, \; -h_1 < x_2 < -h_0\} \tag{3.1.34}$$

and the function W_n^ϵ is the solution of the problem

$$\begin{cases} \dfrac{\partial^2 W_n^\epsilon}{\partial x_1^2} + \dfrac{1}{\epsilon^2}\dfrac{\partial^2 W_n^\epsilon}{\partial x_2^2} = 0 \quad \text{in } Q_n^\epsilon, \\ \dfrac{\partial W_n^\epsilon}{\partial \nu} = 0 \quad \text{on } \partial Q_n^\epsilon \backslash \Gamma_n^\epsilon, \\ W_n^\epsilon(x_1, x_2) = \phi(x_1) \quad \text{on } \Gamma_n^\epsilon, \end{cases}$$

where Γ_n^ϵ is the top of the rectangle, that is,

$$\Gamma_n^\epsilon = \{(x_1, -h_0) : \gamma_{n,\epsilon} \le x_1 \le \gamma_{n+1,\epsilon}\}.$$

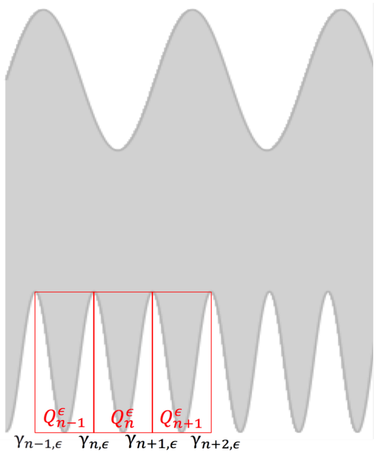


Figure 3.4: Rectangles Q_n^ϵ

From Lemma 3.1.4 we have

$$\left\| \frac{\partial W_n^\epsilon}{\partial x_1} \right\|^2_{L^2(Q_n^\epsilon)} + \frac{1}{\epsilon^2} \left\| \frac{\partial W_n^\epsilon}{\partial x_2} \right\|^2_{L^2(Q_n^\epsilon)} \le C\epsilon^{\alpha-1} \|\phi'\|^2_{L^2(\gamma_{n,\epsilon},\gamma_{n+1,\epsilon})}, \tag{3.1.35}$$

for some constant C independent of ϵ, n and ϕ.

Observe that if we denote $Q^\epsilon = \cup_{n=0}^{N_\epsilon} Q_n^\epsilon$, we get $\widetilde{\Omega}_-^\epsilon = Q^\epsilon \cap \widetilde{\Omega}^\epsilon$. Then, we define the function $W^\epsilon \in H^1(\widetilde{\Omega}_-^\epsilon)$ by

$$W^\epsilon(x_1, x_2) = W_n^\epsilon(x_1, x_2) \quad \text{ as } (x_1, x_2) \in Q_n^\epsilon \cap \widetilde{\Omega}_-^\epsilon,$$

which satisfies the following inequality

$$\begin{aligned} \left\| \frac{\partial W^\epsilon}{\partial x_1} \right\|^2_{L^2(\widetilde{\Omega}_-^\epsilon)} + \frac{1}{\epsilon^2} \left\| \frac{\partial W^\epsilon}{\partial x_2} \right\|^2_{L^2(\widetilde{\Omega}_-^\epsilon)} &\le \sum_{i=0}^{N_\epsilon} \left(\left\| \frac{\partial W_n^\epsilon}{\partial x_1} \right\|^2_{L^2(Q_n^\epsilon)} + \frac{1}{\epsilon^2} \left\| \frac{\partial W_n^\epsilon}{\partial x_2} \right\|^2_{L^2(Q_n^\epsilon)} \right) \\ &\le \sum_{i=0}^{N_\epsilon} C\,\epsilon^{\alpha-1} \, \|\phi'\|^2_{L^2(\gamma_{n,\epsilon},\gamma_{n+1,\epsilon})} \\ &\le C\,\epsilon^{\alpha-1} \, \|\phi'\|^2_{L^2(0,1)} \, . \end{aligned} \tag{3.1.36}$$

Hence, we can rewrite φ^ϵ as

$$\varphi^\epsilon(x_1, x_2) = \begin{cases} W^\epsilon(x_1, x_2), & (x_1, x_2) \in \widetilde{\Omega}_-^\epsilon, \\ \phi(x_1), & (x_1, x_2) \in \Omega_0. \end{cases} \tag{3.1.37}$$

Moreover, following the same arguments as those we used in (3.1.28) we can prove that

$$\|\varphi^\epsilon - \phi\|_{L^2(\widetilde{\Omega}^\epsilon)} \xrightarrow{\epsilon \to 0} 0. \tag{3.1.38}$$

Indeed, since

$$\left|\varphi^\epsilon(x_1,x_2)-\phi(x_1)\right| = \left|\varphi^\epsilon(x_1,x_2)-\varphi^\epsilon(x_1,0)\right| = \left|\int_0^{x_2}\frac{\partial\varphi^\epsilon}{\partial x_2}(x_1,s)\,ds\right|,$$

Then, it follows from (3.1.37) and (3.1.36) that

$$\begin{aligned}
\|\varphi^\epsilon-\phi\|^2_{L^2(\widetilde{\Omega}^\epsilon)} &\le (g_1+h_1)^2\left\|\frac{\partial\varphi^\epsilon}{\partial x_2}\right\|^2_{L^2(\widetilde{\Omega}^\epsilon)}\\
&\le (g_1+h_1)^2\left\|\frac{\partial W^\epsilon}{\partial x_2}\right\|^2_{L^2(\widetilde{\Omega}^\epsilon_-)}\\
&\le (g_1+h_1)^2\,\epsilon^{1+\alpha}\,\|\phi'\|^2_{L^2(0,1)} \overset{\epsilon\to 0}{\longrightarrow} 0.
\end{aligned}$$

Thus, we obtain convergence (3.1.38).

(e) Passing to the limit in the weak formulation.

We can now pass to the limit in (3.1.25) by making use of the test functions φ^ϵ defined above. For this, we study the convergence of each term in (3.1.25).

- First integrand:

$$\int_{\Omega_0}\left\{\widetilde{\frac{\partial u^\epsilon}{\partial x_1}}\frac{\partial\varphi^\epsilon}{\partial x_1}+\frac{1}{\epsilon^2}\widetilde{\frac{\partial u^\epsilon}{\partial x_2}}\frac{\partial\varphi^\epsilon}{\partial x_2}\right\}dx_1dx_2 \overset{\epsilon\to 0}{\longrightarrow}\int_{\Omega_0}\xi^*(x_1,x_2)\phi'(x_1)\,dx_1dx_2. \tag{3.1.39}$$

Thanks to the choice of the test function (3.1.37), for all $\epsilon>0$ one has

$$\left.\frac{\partial\varphi^\epsilon}{\partial x_1}\right|_{\Omega_0}=\frac{\partial\phi}{\partial x_1}=\phi' \quad\text{and}\quad \left.\frac{\partial\varphi^\epsilon}{\partial x_2}\right|_{\Omega_0}=\frac{\partial\phi}{\partial x_2}=0.$$

Consequently,

$$\int_{\Omega_0}\left\{\widetilde{\frac{\partial u^\epsilon}{\partial x_1}}\frac{\partial\varphi^\epsilon}{\partial x_1}+\frac{1}{\epsilon^2}\widetilde{\frac{\partial u^\epsilon}{\partial x_2}}\frac{\partial\varphi^\epsilon}{\partial x_2}\right\}dx_1dx_2=\int_{\Omega_0}\widetilde{\frac{\partial u^\epsilon}{\partial x_1}}(x_1,x_2)\,\phi'(x_1)\,dx_1dx_2.$$

From (3.1.29) we can pass to the limit in the right-hand side to obtain (3.1.39).

- Second integrand:

$$\int_{\widetilde{\Omega}^\epsilon_-}\left\{\frac{\partial u^\epsilon}{\partial x_1}\frac{\partial\varphi^\epsilon}{\partial x_1}+\frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial\varphi^\epsilon}{\partial x_2}\right\}dx_1dx_2 \overset{\epsilon\to 0}{\longrightarrow} 0. \tag{3.1.40}$$

From the definition of φ^ϵ we can write

$$\int_{\widetilde{\Omega}^\epsilon_-}\left\{\frac{\partial u^\epsilon}{\partial x_1}\frac{\partial\varphi^\epsilon}{\partial x_1}+\frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial\varphi^\epsilon}{\partial x_2}\right\}dx_1dx_2=\int_{\widetilde{\Omega}^\epsilon_-}\left\{\frac{\partial u^\epsilon}{\partial x_1}\frac{\partial W^\epsilon}{\partial x_1}+\frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial W^\epsilon}{\partial x_2}\right\}dx_1dx_2 \tag{3.1.41}$$

From Cauchy-Schwarz inequality, (3.1.24) and (3.1.36) we have

$$\left| \int_{\widetilde{\Omega}^\epsilon_-} \Big\{ \frac{\partial u^\epsilon}{\partial x_1} \frac{\partial \varphi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2} \frac{\partial \varphi^\epsilon}{\partial x_2} \Big\} dx_1 dx_2 \right|$$

$$= \left| \int_{\widetilde{\Omega}^\epsilon_-} \Big\{ \frac{\partial u^\epsilon}{\partial x_1} \frac{\partial W^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2} \frac{\partial W^\epsilon}{\partial x_2} \Big\} dx_1 dx_2 \right|$$

$$\leq \left(\int_{\widetilde{\Omega}^\epsilon_-} \Big\{ \Big(\frac{\partial u^\epsilon}{\partial x_1}\Big)^2 + \frac{1}{\epsilon^2} \Big(\frac{\partial u^\epsilon}{\partial x_2}\Big)^2 \Big\} dx_1 dx_2 \right)^{1/2} \left(\int_{\widetilde{\Omega}^\epsilon_-} \Big\{ \Big(\frac{\partial W^\epsilon}{\partial x_1}\Big)^2 + \frac{1}{\epsilon^2} \Big(\frac{\partial W^\epsilon}{\partial x_2}\Big)^2 \Big\} dx_1 dx_2 \right)^{1/2}$$

$$\leq C\, \epsilon^{(\alpha-1)/2} \, \|\phi'\|_{L^2(0,1)}.$$

Therefore, taking into account (3.1.24) and $\alpha > 1$ we get the desired convergence.

- Third integrand:

$$\int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon P_\epsilon u^\epsilon \, \varphi^\epsilon \, dx_1 dx_2 \overset{\epsilon \to 0}{\longrightarrow} \int_0^1 \Big(\frac{|Y^*|}{L_1} + p\Big) \, u_0(x_1) \, \phi(x_1) \, dx_1, \qquad (3.1.42)$$

where p is the constant defined in (3.1.18).

For this, note that we can rewrite the integral of the left-hand side of (3.1.42) as

$$\int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon P_\epsilon u^\epsilon \, \varphi^\epsilon \, dx_1 dx_2 = \int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon \, (P_\epsilon u^\epsilon - u_0) \, \varphi^\epsilon \, dx_1 dx_2$$
$$+ \int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon u_0 \, (\varphi^\epsilon - \phi) \, dx_1 dx_2 + \int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon u_0 \, \phi \, dx_1 dx_2. \qquad (3.1.43)$$

Now we pass to the limit on the right-hand side in order to obtain (3.1.42). From (3.1.28) and (3.1.38), we have that the first two terms tend to zero

$$\int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon \, (P_\epsilon u^\epsilon - u_0) \, \varphi^\epsilon \, dx_1 dx_2 \overset{\epsilon \to 0}{\longrightarrow} 0 \text{ and } \int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon u_0 \, (\varphi^\epsilon - \phi) \, dx_1 dx_2 \overset{\epsilon \to 0}{\longrightarrow} 0. \qquad (3.1.44)$$

Moreover, we may rewrite the third term as follows

$$\int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon u_0 \, \phi \, dx_1 dx_2 = \int_{\widetilde{\Omega}^\epsilon_-} u_0 \, \phi \, dx_1 dx_2 + \int_{\Omega_0} \chi^\epsilon u_0 \, \phi \, dx_1 dx_2$$
$$= \int_0^1 u_0 \, \phi \left(h\left(\frac{x_1}{\epsilon^\alpha}\right) - h_0 \right) \, dx_1 + \int_{\Omega_0} \chi^\epsilon u_0 \, \phi \, dx_1 dx_2.$$

Therefore, from (3.1.6), (3.1.31) and (3.1.32) we get

$$\int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon u_0 \, \phi \, dx_1 dx_2 \to \int_0^1 u_0(x_1) \phi(x_1) \frac{1}{L_2} \left(\int_0^{L_2} h(s) \, ds - h_0 \right) dx_1$$

$$+\int_{\Omega_0} \theta(x_2)\, u_0(x_1)\, \phi(x_1)\, dx_1 dx_2 = \int_0^1 (\frac{|Y^*|}{L_1} + p) u_0(x_1)\, \phi(x_1)\, dx_1 dx_2 \tag{3.1.45}$$

Hence, taking into account (3.1.44), (3.1.45) and (3.1.43) we obtain the requested convergence.

- Fourth integrand:

$$\int_{\widetilde{\Omega}^\epsilon} \widetilde{f}^\epsilon\, \varphi^\epsilon\, dx_1 dx_2 \overset{\epsilon \to 0}{\longrightarrow} \int_0^1 \hat{f}(x_1)\, \phi(x_1)\, dx_1. \tag{3.1.46}$$

From (3.1.38) and the hypotheses of Theorem 3.1.5 we get the convergence (3.1.46)

$$\begin{aligned}\int_{\widetilde{\Omega}^\epsilon} \widetilde{f}^\epsilon\, \varphi^\epsilon\, dx_1 dx_2 &= \int_{\widetilde{\Omega}^\epsilon} \widetilde{f}^\epsilon\, (\varphi^\epsilon - \phi)\, dx_1 dx_2 + \int_0^1 \hat{f}^\epsilon\, \phi\, dx_1 \\ &\overset{\epsilon \to 0}{\longrightarrow} \int_0^1 \hat{f}(x_1)\, \phi(x_1)\, dx_1.\end{aligned}$$

Consequently, we can pass to the limit in (3.1.25) taking φ^ϵ as test function. Therefore, using (3.1.39), (3.1.40), (3.1.42) and (3.1.46), we obtain the following limit variational formulation:

$$\begin{aligned}&\int_{\Omega_0} \Big\{ \xi^*(x_1, x_2)\, \phi'(x_1) + (\frac{|Y^*|}{L_1} + p)\, u_0(x_1)\, \phi(x_1) \Big\}\, dx_2 dx_1 \\ &\quad = \int_0^1 \hat{f}(x_1)\, \phi(x_1)\, dx_1, \ \forall \phi \in H^1(0,1).\end{aligned} \tag{3.1.47}$$

At this point the question is how to relate u_0 to ξ^*. In the following paragraphs we obtain the equation satisfied by ξ^* adapting the Tartar's method of oscillating test functions to the case of thin domains with resonant and fast oscillations. The approach relies on the construction of a class of oscillating test functions which allows us to pass to the limit eliminating all terms containing a product of two weakly convergent sequences. We will accomplish this using the solution of the auxiliary problem set in the basic cell, see (3.1.20), and the solutions of certain problems defined in the rectangles Q_n^ϵ, see (3.1.34).

(f) Relation between ξ^* and u_0.

Let us consider the following families of diffeomorphisms $T_k^\epsilon : A_k^\epsilon \mapsto Y$ given by

$$T_k^\epsilon(x_1, x_2) = \Big(\frac{x_1 - \epsilon k L_1}{\epsilon}, x_2 \Big), \tag{3.1.48}$$

where

$$A_k^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid \epsilon k L_1 \le x_1 < \epsilon L_1(k+1), -h_0 < x_2 < g_1\}, \quad \forall k \in \mathbb{N},$$

$$Y = (0, L_1) \times (-h_0, g_1).$$

Moreover, we consider the extension operator

$$P \in \mathcal{L}(H^1(Y^*), H^1(Y)) \cap \mathcal{L}(L^2(Y^*), L^2(Y)),$$

defined using the same reflection procedure as Lemma 3.1.2, see Remark 3.1.3.

Then, using this extension operator, the diffeomorphisms (3.1.48) and the unique solution of the auxiliary problem (3.1.20) we define ω_k^ϵ in $(x_1, x_2) \in A_k^\epsilon$ by

$$\omega_k^\epsilon(x_1, x_2) \quad = \quad x_1 - \epsilon\Big(PX \circ T_k^\epsilon(x_1, x_2)\Big) = x_1 - \epsilon\Big(PX\Big(\frac{x_1 - \epsilon L_1 k}{\epsilon}, x_2\Big)\Big).$$

Observe that for any $(x_1, x_2) \in \Omega_0$ there exists $k \in \mathbb{N}$ such that $(x_1, x_2) \in A_k^\epsilon$. Therefore, the function $\omega^\epsilon(x_1, x_2) = \omega_k^\epsilon(x_1, x_2)$ is well defined and $\omega^\epsilon \in H^1(\Omega_0)$.

We introduce now the vector $\eta^\epsilon = (\eta_1^\epsilon, \eta_2^\epsilon)$ defined by

$$\eta_i^\epsilon(x_1, x_2) = \frac{\partial \omega^\epsilon}{\partial x_i}(x_1, x_2), \quad (x_1, x_2) \in \Omega_+^\epsilon, \tag{3.1.49}$$

where

$$\Omega_+^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ 0 < x_1 < 1 \text{ and } - h_0 < x_2 < g(x_1/\epsilon)\}.$$

Note that $\Omega_+^\epsilon = \Omega_0 \cap \Omega^\epsilon$.

Then, taking into account the definition of X, see (3.1.20), we obtain that

$$\begin{cases} \dfrac{\partial \eta_1^\epsilon}{\partial x_1} + \dfrac{1}{\epsilon^2}\dfrac{\partial \eta_2^\epsilon}{\partial x_2} = 0 \text{ in } \Omega_+^\epsilon, \\ \eta_1^\epsilon N_1^\epsilon + \dfrac{1}{\epsilon^2}\eta_2^\epsilon N_2^\epsilon = 0 \text{ on } \Big(x_1, g\Big(\dfrac{x_1}{\epsilon}\Big)\Big), \\ \eta_1^\epsilon N_1^\epsilon + \dfrac{1}{\epsilon^2}\eta_2^\epsilon N_2^\epsilon = 0 \text{ on } (x_1, -h_0), \end{cases} \tag{3.1.50}$$

where

$$N^\epsilon = (N_1^\epsilon, N_2^\epsilon) = \left(-\frac{g'(\frac{x_1}{\epsilon})}{(\epsilon^2 + g'(\frac{x_1}{\epsilon})^2)^{\frac{1}{2}}}, \frac{\epsilon}{(\epsilon^2 + g'(\frac{x_1}{\epsilon})^2)^{\frac{1}{2}}} \right) \text{ on } \Big(x_1, g\Big(\frac{x_1}{\epsilon}\Big)\Big),$$

$$N^\epsilon = (0, -1) \text{ on } (x_1, -h_0).$$

Therefore, multiplying (3.1.50) by a test function $\psi \in H^1(\Omega_+^\epsilon)$ with $\psi = 0$ in a neighborhood of the lateral boundaries and integrating by parts, we get

$$\int_{\Omega_+^\epsilon} \left(\eta_1^\epsilon \frac{\partial \psi}{\partial x_1} + \eta_2^\epsilon \frac{1}{\epsilon^2}\frac{\partial \psi}{\partial x_2} \right) dx_1 dx_2 = 0. \tag{3.1.51}$$

Thus, from the variational formulation (3.1.25) and identity (3.1.51) we have

$$\int_{\widetilde{\Omega}^\epsilon} \Big\{ \widetilde{\frac{\partial u^\epsilon}{\partial x_1}} \frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2} \widetilde{\frac{\partial u^\epsilon}{\partial x_2}} \frac{\partial \varphi}{\partial x_2} \Big\} dx_1 dx_2 + \int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon P_\epsilon u^\epsilon \varphi dx_1 dx_2$$

$$-\int_{\Omega_+^\epsilon}\left(\eta_1^\epsilon\frac{\partial\psi}{\partial x_1}+\eta_2^\epsilon\frac{1}{\epsilon^2}\frac{\partial\psi}{\partial x_2}\right)dx_1dx_2=\int_{\widetilde{\Omega}^\epsilon}\chi^\epsilon f^\epsilon\varphi dx_1dx_2, \qquad (3.1.52)$$

for any $\varphi\in H^1(\Omega^\epsilon)$ and $\psi\in H^1(\Omega_+^\epsilon)$ with $\psi=0$ in a neighborhood of the lateral boundaries of Ω_+^ϵ.

As we have mentioned, we will construct appropriate test functions, which used in the identity (3.1.52) allow us to pass to the limit in all the terms.

From now on M^ϵ denotes the integer part of $\frac{1}{\epsilon L_1}$ for each $\epsilon>0$. Observe that $M^\epsilon\sim L_1^{-1}\epsilon^{-1}$.

(f.1) Limit of ω^ϵ.

From the definition of ω^ϵ and using the properties of the extension operator P, we have by a simple change of variables that

$$\int_{A_k^\epsilon}|\omega^\epsilon-x_1|^2dx_1dx_2=\int_Y\epsilon^3|(PX)(y_1,y_2)|^2dy_1dy_2\le C\epsilon^3\int_{Y^*}|X(y_1,y_2)|^2dy_1dy_2.$$

Therefore, taking into account (3.1.48) we obtain

$$\begin{aligned}\int_{\Omega_0}|\omega^\epsilon-x_1|^2dx_1dx_2&\le\sum_{k=0}^{M^\epsilon}\int_{A_k^\epsilon}|\omega^\epsilon-x_1|^2dx_1dx_2\le\sum_{k=0}^{M^\epsilon}C\epsilon^3\int_{Y^*}|X(y_1,y_2)|^2dy_1dy_2\\&\le C\epsilon^2\int_{Y^*}|X(y_1,y_2)|^2dy_1dy_2\overset{\epsilon\to0}{\longrightarrow}0.\end{aligned}$$

With a very similar argument, we can prove that

$$\int_{\Omega_0}\left|\frac{\partial}{\partial x_1}\left(\omega^\epsilon-x_1\right)\right|^2dx_1dx_2\le C\int_{Y^*}\left|\frac{\partial X}{\partial y_1}(y_1,y_2)\right|^2dy_1dy_2,\quad\forall\epsilon>0.$$
$$\int_{\Omega_0}\left|\frac{\partial}{\partial x_2}\left(\omega^\epsilon-x_1\right)\right|^2dx_1dx_2\le C\epsilon^2\int_{Y^*}\left|\frac{\partial X}{\partial y_2}(y_1,y_2)\right|^2dy_1dy_2\overset{\epsilon\to0}{\longrightarrow}0.$$

Then, we can conclude

$$\omega^\epsilon\overset{\epsilon\to0}{\longrightarrow}x_1\quad s-L^2(\Omega_0)\text{ and }w-H^1(\Omega_0), \qquad (3.1.53)$$

and

$$\frac{\partial\omega^\epsilon}{\partial x_2}\overset{\epsilon\to0}{\longrightarrow}0\quad s-L^2(\Omega_0). \qquad (3.1.54)$$

(f.2) Limit of η_1^ϵ.

From definition (3.1.49) and the L_1 periodicity of the function X we have that

$$\eta_1^\epsilon(x_1,x_2)=1-\frac{\partial X}{\partial y_1}\Big(\frac{x_1-\epsilon kL_1}{\epsilon},x_2\Big)=1-\frac{\partial X}{\partial y_1}\Big(\frac{x_1}{\epsilon},x_2\Big).$$

Consequently, if $\widetilde{\eta}^\epsilon=\eta^\epsilon\chi^\epsilon$ denotes the extension by zero of the vector η^ϵ to the whole Ω_0 by the Average Theorem (see, e.g., [52, p. xvi])

$$\widetilde{\eta}_1^\epsilon(x_1,x_2)\overset{\epsilon\to0}{\rightharpoonup}\frac{1}{L_1}\int_0^{L_1}\Big(1-\frac{\widetilde{\partial X}}{\partial y_1}(s,x_2)\Big)\chi(s,x_2)ds:=q(x_2)\quad w^*-L^\infty(0,1) \qquad (3.1.55)$$

where χ is the characteristic function of Y^*.

Hence, arguing as (3.1.31), by Lebesgue's Dominated Convergence Theorem we obtain

$$\widetilde{\eta}_1^\epsilon \overset{\epsilon\to 0}{\rightharpoonup} q \quad w^* - L^\infty(\Omega_0). \tag{3.1.56}$$

(f.3) Test function.

Let $\phi = \phi(x_1) \in \mathcal{C}_0^\infty(0,1)$. Using the notation established in paragraph **(d)** we introduce the test function

$$\psi^\epsilon(x_1,x_2) = \begin{cases} V_n^\epsilon(x_1,x_2), & (x_1,x_2) \in \widetilde{\Omega}_-^\epsilon \cap Q_n^\epsilon, \quad n = 0,1,\dots \\ \phi(x_1)\omega^\epsilon(x_1,x_2), & (x_1,x_2) \in \Omega_0, \end{cases} \tag{3.1.57}$$

where ω^ϵ is defined above and, as in (3.1.34), Q_n^ϵ is the rectangle

$$Q_n^\epsilon = \{(x_1,x_2) \,|\, \gamma_{n,\epsilon} < x_1 < \gamma_{n+1,\epsilon}, \; -h_1 < x_2 < -h_0\}$$

and the function V_n^ϵ is the solution of the problem

$$\begin{cases} \dfrac{\partial^2 V_n^\epsilon}{\partial x_1^2} + \dfrac{1}{\epsilon^2}\dfrac{\partial^2 V_n^\epsilon}{\partial x_2^2} = 0, \quad \text{in } Q_n^\epsilon \\ \dfrac{\partial V_n^\epsilon}{\partial \nu} = 0, \quad \text{on } \partial Q_n^\epsilon \backslash \Gamma_n^\epsilon \\ V_n^\epsilon(x_1,x_2) = \phi(x_1)\omega^\epsilon(x_1,-h_0), \quad \text{on } \Gamma_n^\epsilon \end{cases}$$

where Γ_n^ϵ is the top of the rectangle, that is,

$$\Gamma_n^\epsilon = \{(x_1,-h_0) : \gamma_{n,\epsilon} \le x_1 \le \gamma_{n+1,\epsilon}\}.$$

Now, we want to show that

$$\|\psi^\epsilon - \phi x_1\|_{L^2(\widetilde{\Omega}^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0.$$

Since $\widetilde{\Omega}_-^\epsilon = \cup_{n=0}^{N_\epsilon} Q_n^\epsilon \cap \widetilde{\Omega}^\epsilon$ we define the function $V^\epsilon \in H^1(\widetilde{\Omega}_-^\epsilon)$ by

$$V^\epsilon(x_1,x_2) = V_n^\epsilon(x_1,x_2) \quad \text{as } (x_1,x_2) \in Q_n^\epsilon \cap \widetilde{\Omega}_-^\epsilon.$$

From Lemma 3.1.4 we have that the function V^ϵ satisfies the following estimate

$$\left\|\frac{\partial V^\epsilon}{\partial x_1}\right\|^2_{L^2(\widetilde{\Omega}_{-1}^\epsilon)} + \frac{1}{\epsilon^2}\left\|\frac{\partial V^\epsilon}{\partial x_2}\right\|^2_{L^2(\widetilde{\Omega}_{-1}^\epsilon)} \le C\,\epsilon^{\alpha-1}\left\|\frac{\partial\big(\phi(x_1)\omega^\epsilon(x_1,-h_0)\big)}{\partial x_1}\right\|^2_{L^2(0,1)} \tag{3.1.58}$$

where C denotes a constant independent of ϵ. Moreover, notice that

$$\omega^\epsilon(x_1,-h_0) \in H^1(0,1),$$

Indeed, recall that ω^ϵ is defined as

$$\omega_k^\epsilon(x_1,x_2) = x_1 - \epsilon\Big(PX\Big(\frac{x_1 - \epsilon L_1 k}{\epsilon}, x_2\Big)\Big), \quad \text{a.e. } (x_1,x_2) \in \Omega_0 \cap A_k^\epsilon.$$

Then, since from standard elliptic regularity theory $X \in H^2(Y^*)$ we can ensure that $\omega^\epsilon(x_1, -h_0) \in H^1(0,1)$.

Thus, using the properties of ω^ϵ and in view of (3.1.58) we prove that

$$\left\| \frac{\partial V^\epsilon}{\partial x_1} \right\|^2_{L^2(\widetilde{\Omega}^\epsilon_{-1})} + \frac{1}{\epsilon^2} \left\| \frac{\partial V^\epsilon}{\partial x_2} \right\|^2_{L^2(\widetilde{\Omega}^\epsilon_{-1})} \le C\, \epsilon^{\alpha-1}. \tag{3.1.59}$$

To do so, taking into account (3.1.58) we only need to prove the following inequality

$$\left\| \frac{\partial\big(\phi(x_1)\omega^\epsilon(x_1, -h_0)\big)}{\partial x_1} \right\|^2_{L^2(0,1)} \le C. \tag{3.1.60}$$

Observe that

$$\begin{aligned} \left\| \frac{\partial\big(\phi(x_1)\omega^\epsilon(x_1, -h_0)\big)}{\partial x_1} \right\|_{L^2(0,1)} &\le C \left\| \omega^\epsilon(x_1, -h_0) \right\|_{H^1(0,1)} \\ &\le C \left\| \omega^\epsilon(x_1, -h_0) - x_1 \right\|_{H^1(0,1)} + C \left\| x_1 \right\|_{H^1(0,1)}. \end{aligned}$$

Consequently, we obtain (3.1.60) taking into account that

$$\begin{aligned} \|\omega^\epsilon(x_1, -h_0) - x_1\|^2_{H^1(0,1)} &\le \sum_{k=1}^{M^\epsilon+1} \int_{\epsilon(k-1)L_1}^{\epsilon k L_1} |\omega^\epsilon(x_1, -h_0) - x_1|^2 \, dx_1 \\ &\quad + \int_{\epsilon(k-1)L_1}^{\epsilon k L_1} \Big| \frac{\partial}{\partial x_1} (\omega^\epsilon(x_1, -h_0) - x_1) \Big|^2 dx_1 \\ &\le \sum_{k=1}^{M^\epsilon+1} \int_0^{L_1} \Big\{ \epsilon^3 |X(y_1, -h_0)|^2 + \epsilon \Big| \frac{\partial X}{\partial y_1}(y_1, -h_0) \Big|^2 \Big\} dy_1 \\ &\le C \int_0^{L_1} \Big\{ \epsilon^2 |X(y_1, -h_0)|^2 + \Big| \frac{\partial X}{\partial y_1}(y_1, -h_0) \Big|^2 \Big\} dy_1 \le C. \end{aligned}$$

Therefore, we get the estimate (3.1.60) and, as a consequence, we have (3.1.59).

Now, we can argue as in (3.1.38) to show

$$\|\psi^\epsilon - \phi \overline{P} \omega^\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0. \tag{3.1.61}$$

where $\overline{P}\omega^\epsilon$ is the function defined on the set

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ -h_1 < x_2 < g_1\},$$

which is obtained from ω^ϵ using the extension operator $\overline{P}$ defined by reflection in the negative vertical direction along the line $x_2 = -h_0$.

Indeed, observe that since $\overline{P}\omega^\epsilon$ is defined by reflection in the negative vertical direction along the line $x_2 = -h_0$ we have that given any $(x_1, x_2) \in \widetilde{\Omega}^\epsilon_-$ there exists $(x_1, \widetilde{x_2}) \in \Omega_0$ such that

$$\psi^\epsilon(x_1, x_2) - \phi(x_1)\overline{P}\omega^\epsilon(x_1, x_2) = \psi^\epsilon(x_1, x_2) - \phi(x_1)\omega^\epsilon(x_1, \widetilde{x_2})$$

$$
\begin{aligned}
&= \psi^\epsilon(x_1, x_2) - \psi^\epsilon(x_1, \widetilde{x_2}) \\
&= -\int_{x_2}^{\widetilde{x_2}} \frac{\partial \psi^\epsilon}{\partial x_2}(x_1, s)ds, \quad \forall\,(x_1, x_2) \in \widetilde{\Omega}_-^\epsilon.
\end{aligned}
$$

Moreover, since $\psi^\epsilon(x_1, x_2) - \phi(x_1)\overline{P}\omega^\epsilon(x_1, x_2) = 0\ \forall\,(x_1, x_2) \in \Omega_0$, we have by the equality above that

$$
\begin{aligned}
\|\psi^\epsilon - \phi\overline{P}\omega^\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)} &\leq C(g, h) \left\| \frac{\partial \psi^\epsilon}{\partial x_2} \right\|^2_{L^2(\widetilde{\Omega}^\epsilon)} \\
&\leq C(g, h) \left\| \frac{\partial V^\epsilon}{\partial x_2} \right\|^2_{L^2(\widetilde{\Omega}_-^\epsilon)} + C(g, h) \left\| \frac{\partial \omega^\epsilon}{\partial x_2}\phi \right\|^2_{L^2(\Omega_0)}.
\end{aligned}
$$

Therefore, by (3.1.54) and (3.1.59) we conclude that ψ^ϵ verifies (3.1.61).

Consequently, since

$$
\begin{aligned}
\|\psi^\epsilon - \phi x_1\|_{L^2(\widetilde{\Omega}^\epsilon)} &\leq \|\psi^\epsilon - \phi\overline{P}\omega^\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)} + \|\phi\overline{P}\omega^\epsilon - \phi x_1\|_{L^2(\widetilde{\Omega}^\epsilon)} \\
&\leq \|\psi^\epsilon - \phi\overline{P}\omega^\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)} + C\|\phi\omega^\epsilon - \phi x_1\|_{L^2(\Omega_0)},
\end{aligned}
$$

from (3.1.61) and (3.1.53) we get

$$
\|\psi^\epsilon - \phi x_1\|_{L^2(\widetilde{\Omega}^\epsilon)} \xrightarrow{\epsilon \to 0} 0. \tag{3.1.62}
$$

(f.4) Passing to the limit.

Now we pass to the limit in the equality (3.1.52) with the test functions $\varphi = \psi^\epsilon$ defined in (3.1.57) and $\psi = \phi u^\epsilon$. We rewrite (3.1.52) as

$$
\begin{aligned}
&\int_{\widetilde{\Omega}_-^\epsilon} \left\{ \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \psi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial \psi^\epsilon}{\partial x_2} \right\} dx_1 dx_2 - \int_{\Omega_+^\epsilon} \left(\eta_1^\epsilon \frac{\partial(\phi u^\epsilon)}{\partial x_1} + \eta_2^\epsilon \frac{1}{\epsilon^2}\frac{\partial(\phi u^\epsilon)}{\partial x_2} \right) dx_1 dx_2 \\
&+ \int_{\Omega_0} \left\{ \widetilde{\frac{\partial u^\epsilon}{\partial x_1}}\frac{\partial \psi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2}\widetilde{\frac{\partial u^\epsilon}{\partial x_2}}\frac{\partial \psi^\epsilon}{\partial x_2} \right\} dx_1 dx_2 + \int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon P_\epsilon u^\epsilon \psi^\epsilon dx_1 dx_2 \\
&= \int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon f^\epsilon \psi^\epsilon dx_1 dx_2,
\end{aligned} \tag{3.1.63}
$$

- First integrand:

$$
\int_{\widetilde{\Omega}_-^\epsilon} \left\{ \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \psi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial \psi^\epsilon}{\partial x_2} \right\} dx_1 dx_2 \xrightarrow{\epsilon \to 0} 0. \tag{3.1.64}
$$

Taking into account the definition of ψ^ϵ, see (3.1.57), we have

$$
\int_{\widetilde{\Omega}_-^\epsilon} \left\{ \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \psi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial \psi^\epsilon}{\partial x_2} \right\} dx_1 dx_2 = \int_{\widetilde{\Omega}_-^\epsilon} \left\{ \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial V^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial V^\epsilon}{\partial x_2} \right\} dx_1 dx_2.
$$

From Cauchy-Schwarz inequality, (3.1.59) and (3.1.24) we obtain

$$\Big| \int_{\widetilde{\Omega}^\epsilon_-} \Big\{ \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial V^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial V^\epsilon}{\partial x_2} \Big\} dx_1 dx_2 \Big|$$

$$\leq \Bigg(\int_{\widetilde{\Omega}^\epsilon_-} \Big\{ \Big(\frac{\partial u^\epsilon}{\partial x_1}\Big)^2 + \frac{1}{\epsilon^2}\Big(\frac{\partial u^\epsilon}{\partial x_2}\Big)^2 \Big\} dx_1 dx_2 \Bigg)^{1/2} \Bigg(\int_{\widetilde{\Omega}^\epsilon_-} \Big\{ \Big(\frac{\partial V^\epsilon}{\partial x_1}\Big)^2 + \frac{1}{\epsilon^2}\Big(\frac{\partial V^\epsilon}{\partial x_2}\Big)^2 \Big\} dx_1 dx_2 \Bigg)^{1/2}$$

$$\leq C\, \epsilon^{(\alpha-1)/2} \overset{\epsilon\to 0}{\longrightarrow} 0.$$

Hence, we get the desired convergence.

- Second and third integrand:

$$\int_{\Omega_0} \Big\{ \widetilde{\frac{\partial u^\epsilon}{\partial x_1}}\frac{\partial \psi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2}\widetilde{\frac{\partial u^\epsilon}{\partial x_2}}\frac{\partial \psi^\epsilon}{\partial x_2} \Big\} dx_1 dx_2 - \int_{\Omega^\epsilon_+} \Big\{ \eta_1^\epsilon \frac{\partial(\phi u^\epsilon)}{\partial x_1} + \eta_2^\epsilon \frac{1}{\epsilon^2}\frac{\partial(\phi u^\epsilon)}{\partial x_2} \Big\} dx_1 dx_2$$
$$\overset{\epsilon\to 0}{\longrightarrow} \int_{\Omega_0} \Big\{ \xi^* \frac{\partial \phi}{\partial x_1}\, x_1 - q \frac{\partial \phi}{\partial x_1} u_0 \Big\}. \tag{3.1.65}$$

From the definition of ψ^ϵ, see (3.1.57), and using $\eta_i^\epsilon = \frac{\partial \omega^\epsilon}{\partial x_i}, i = 1, 2$, we obtain

$$\begin{aligned}
&\int_{\Omega_0} \Big\{ \widetilde{\frac{\partial u^\epsilon}{\partial x_1}}\frac{\partial \psi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2}\widetilde{\frac{\partial u^\epsilon}{\partial x_2}}\frac{\partial \psi^\epsilon}{\partial x_2} \Big\} dx_1 dx_2 \\
&\quad = \int_{\Omega^\epsilon_+} \Big\{ \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \phi}{\partial x_1}\omega^\epsilon + \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \omega^\epsilon}{\partial x_1}\phi + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial \omega^\epsilon}{\partial x_2}\phi \Big\} dx_1 dx_2 \\
&\quad = \int_{\Omega^\epsilon_+} \Big\{ \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \phi}{\partial x_1}\omega^\epsilon + \frac{\partial u^\epsilon}{\partial x_1}\eta_1^\epsilon \phi + \frac{1}{\epsilon^2}\eta_2^\epsilon \frac{\partial u^\epsilon}{\partial x_2}\phi \Big\} dx_1 dx_2.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\int_{\Omega^\epsilon_+} \Big\{ \eta_1^\epsilon \frac{\partial(\phi u^\epsilon)}{\partial x_1} + \eta_2^\epsilon \frac{1}{\epsilon^2}\frac{\partial(\phi u^\epsilon)}{\partial x_2} \Big\} dx_1 dx_2 \\
&\quad = \int_{\Omega^\epsilon_+} \Big\{ \eta_1^\epsilon \frac{\partial \phi}{\partial x_1} u^\epsilon + \eta_1^\epsilon \frac{\partial u^\epsilon}{\partial x_1}\phi + \eta_2^\epsilon \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\phi \Big\} dx_1 dx_2.
\end{aligned}$$

Therefore, canceling the appropriate terms the second integrand reduces to

$$\int_{\Omega_0} \Big\{ \widetilde{\frac{\partial u^\epsilon}{\partial x_1}}\frac{\partial \psi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2}\widetilde{\frac{\partial u^\epsilon}{\partial x_2}}\frac{\partial \psi^\epsilon}{\partial x_2} \Big\} dx_1 dx_2 - \int_{\Omega^\epsilon_+} \Big\{ \eta_1^\epsilon \frac{\partial(\phi u^\epsilon)}{\partial x_1} + \eta_2^\epsilon \frac{1}{\epsilon^2}\frac{\partial(\phi u^\epsilon)}{\partial x_2} \Big\} dx_1 dx_2$$
$$= \int_{\Omega_0} \Big\{ \widetilde{\frac{\partial u^\epsilon}{\partial x_1}}\frac{\partial \phi}{\partial x_1}\,\omega^\epsilon - \widetilde{\eta}_1^\epsilon \frac{\partial \phi}{\partial x_1}\, P_\epsilon u^\epsilon \Big\} dx_1 dx_2.$$

Therefore, using convergences (3.1.28), (3.1.29), (3.1.53) and (3.1.55) we have (3.1.65).

- Fourth integrand:

$$\int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon P_\epsilon u^\epsilon\, \psi^\epsilon\, dx_1 dx_2 \overset{\epsilon\to 0}{\longrightarrow} \int_0^1 (\frac{|Y^*|}{L_1} + p)\, u_0(x_1)\, \phi(x_1)\, x_1\, dx_1 \tag{3.1.66}$$

Following along the lines of the proof of the convergence (3.1.42) and using convergence (3.1.62) we easily find (3.1.66).

- Fifth integrand:

$$\int_{\widetilde{\Omega}^\epsilon} \widetilde{f}^\epsilon\, \psi^\epsilon\, dx_1 dx_2 \overset{\epsilon\to 0}{\longrightarrow} \int_0^1 \hat{f}(x_1)\, \phi(x_1)\, x_1\, dx_1. \tag{3.1.67}$$

Using the same computations as those made to derive (3.1.46) and taking into account convergence (3.1.62) we obtain (3.1.67).

Therefore, by (3.1.64), (3.1.65), (3.1.66) and (3.1.67), we can pass to the limit in (3.1.63). More precisely, we have

$$\begin{aligned}
&\int_{\Omega_0} \Big\{ \xi^*(x_1,x_2) \frac{\partial \phi}{\partial x_1}(x_1)\, x_1 - q(x_2) \frac{\partial \phi}{\partial x_1}(x_1) u_0(x_1) \Big\} dx_1 dx_2 \\
&+ \int_{\Omega_0} (\frac{|Y^*|}{L_1} + p)\, u_0(x_1)\, \phi(x_1)\, x_1\, dx_1 dx_2 = \int_0^1 \hat{f}(x_1)\, \phi(x_1)\, x_1\, dx_1 \quad \forall \phi \in \mathcal{C}_0^\infty(0,1),
\end{aligned} \tag{3.1.68}$$

where p and q are given by

$$p = \frac{1}{L_2} \int_0^{L_2} h(s) ds - h_0, \quad q(x_2) = \frac{1}{L_1} \int_0^{L_1} \Big(1 - \widetilde{\frac{\partial X}{\partial y_1}}(s, x_2) \Big) \chi(s, x_2) ds.$$

Moreover, due to $\xi^* \frac{\partial}{\partial x_1}(\phi\, x_1) = \xi^* x_1 \frac{\partial \phi}{\partial x_1} + \xi^* \phi$, we can rewrite (3.1.68) as

$$\begin{aligned}
\int_{\Omega_0} \Big\{ \xi^*(x_1,x_2) &\frac{\partial}{\partial x_1}(\phi x_1) - \phi(x_1) \xi^*(x_1,x_2) - q(x_2) \frac{\partial \phi}{\partial x_1}(x_1) u_0(x_1) \Big\} dx_1 dx_2 \\
&+ \int_0^1 (\frac{|Y^*|}{L_1} + p)\, u_0(x_1)\, \phi\, x_1\, dx_1 \\
&= \int_0^1 \hat{f}(x_1)\, \phi(x_1)\, x_1\, dx_1, \quad \forall \phi \in \mathcal{C}_0^\infty(0,1).
\end{aligned} \tag{3.1.69}$$

Let $\phi \in H^1(0,1)$ we take ϕx_1 as a test function in (3.1.47). Then,

$$\begin{aligned}
\int_{\Omega_0} \xi^*(x_1,x_2)\, \frac{\partial}{\partial x_1}(\phi x_1) dx_2 dx_1 + \int_0^1 (\frac{|Y^*|}{L_1} + p)\, u_0(x_1)\, \phi(x_1)\, x_1\, dx_1 \\
= \int_0^1 \hat{f}(x_1)\, \phi(x_1)\, x_1\, dx_1.
\end{aligned} \tag{3.1.70}$$

Therefore, combining (3.1.70) and (3.1.69) we have that

$$0 = \int_{\Omega_0} \Big\{\phi(x_1)\xi^*(x_1,x_2) + q(x_2)\frac{\partial \phi}{\partial x_1}(x_1)u_0(x_1)\Big\} dx_1dx_2, \quad \forall \phi \in \mathcal{C}_0^\infty(0,1).$$

Hence, integrating by parts we obtain

$$0 = \int_{\Omega_0} \Big\{\phi(x_1)\xi^*(x_1,x_2) - q(x_2)\frac{\partial u_0}{\partial x_1}(x_1)\phi(x_1)\Big\} dx_1dx_2, \quad \forall \phi \in \mathcal{C}_0^\infty(0,1). \tag{3.1.71}$$

With the definition of $\hat{q}$ given by (3.1.18) and performing an iterated integration in (3.1.71) we obtain

$$\int_0^1 \phi(x_1)\Big(\int_{-h_0}^{g_1} \xi^*(x_1,x_2)dx_2 - \hat{q}\frac{\partial u_0}{\partial x_1}(x_1)\Big)dx_1 = 0, \qquad \forall\, \phi \,\in\, \mathcal{C}_0^\infty(0,1).$$

So, the function ξ^* satisfies

$$\int_{-h_0}^{g_1} \xi^*(x_1,x_2)dx_2 = \hat{q}\frac{\partial u_0(x_1)}{\partial x_1}, \qquad \text{a.e. } x_1 \in (0,1). \tag{3.1.72}$$

(g) Homogenized limit problem.

Once we have identified ξ^* we are in conditions to obtain the homogenized limit problem. Placing (3.1.72) in the limit equation (3.1.47), we get for all $\phi \in H^1(0,1)$ that

$$\int_0^1 \Big\{\hat{q}\frac{\partial u_0}{\partial x_1}(x_1)\frac{\partial \phi}{\partial x_1}(x_1) + \Big(\frac{|Y^*|}{L_1} + p\Big)\, u_0(x_1)\,\phi(x_1)\,\Big\}dx_1 = \int_0^1 \hat{f}(x_1)\,\phi(x_1)\,dx_1. \tag{3.1.73}$$

Hence, u_0 is the unique solution of (3.1.73), which is the weak formulation of (3.1.17). Moreover, since any convergent subsequence of $\{u^\epsilon\}$ tends to the same limit u_0, we get that the whole sequence will also converge to u_0. This completes the proof of Theorem 3.1.5. □

Remark 3.1.10. *Observe that problem* (3.1.17) *is well posed in the sense that the diffusion coefficient is strictly greater than zero. Actually, we will show that* $\hat{q} > 0$. *Let* $a(\cdot,\cdot)$ *be the bilinear form associated with the variational formulation of* (3.1.20)

$$a(\Psi,\Phi) = \int_{Y^*} \nabla\Psi\cdot\nabla\Phi\, dy_1dy_2,$$

for $\Psi,\Phi \in H^1_\#(Y^*)$. *Then,* X *satisfies*

$$a(X,\Phi) = \int_{B_1} N_1\Phi\, dS, \text{ for any } \Phi \in H^1_\#(Y^*)$$

where $N_1 = -\frac{g'(y_1)}{\sqrt{1+g'(y_1)^2}}$. *Consequently,*

$$a(y_1 - X, \Phi) = \int_{B_1} N_1 \Phi \, dS - \int_{B_1} N_1 \Phi \, dS = 0, \text{ for any } \Phi \in H^1_\#(Y^*). \qquad (3.1.74)$$

Moreover, we have

$$\hat{q} = \frac{1}{|Y^*|} \int_{Y^*} \Big\{ 1 - \frac{\partial X}{\partial y_1} \Big\} dy_1 dy_2 = \frac{1}{|Y^*|} \int_{Y^*} \frac{\partial}{\partial y_1} (y_1 - X) \frac{\partial y_1}{\partial y_1} dy_1 dy_2 = \frac{1}{|Y^*|} a(y_1 - X, y_1).$$

Hence, using (3.1.74) *we get*

$$|Y^*| \hat{q} = a(y_1 - X, y_1) - a(y_1 - X, -X) = a(y_1 - X, y_1 - X) = \|\nabla(y_1 - X)\|^2_{[L^2(Y^*)]^2}.$$

Therefore, since $|Y^*| > 0$ *we can conclude that* $\hat{q} > 0$. *Indeed, if this is not true, then we would have*

$$\frac{\partial(y_1 - X)}{\partial y_1} = 0, \quad \frac{\partial(y_1 - X)}{\partial y_2} = 0$$

which implies that there exists a constant C *such that* $y_1 - X = C$. *This is impossible because* X *is* L_1*-periodic in the first variable.*

3.1.3. Corrector result in thin domains with resonant and fast oscillations

In this subsection we construct a corrector function to derive a kind of strong convergence in H^1-norm. Then, we introduce a suitable corrector $\kappa^\epsilon \in H^1(R^\epsilon)$, with $\kappa^\epsilon = \mathrm{o}(\epsilon)$ in $L^2(R^\epsilon)$ in order to improve the convergences obtained in Theorem 3.1.5. Actually, we find an explicit corrector term $\kappa^\epsilon \in H^1(R^\epsilon)$ such that

$$\epsilon^{-1/2} ||w^\epsilon - (u_0 + \kappa^\epsilon)||_{H^1(R^\epsilon)} \overset{\epsilon \to 0}{\longrightarrow} 0.$$

where w^ϵ is the solution of (3.0.1).

Since we will obtain the corrector result for the solutions of the problem (3.0.1) posed in the thin domain R^ϵ we first introduce the functional setting for the perturbed problem (3.0.1) and the homogenized limit one (3.1.17).

3.1.3.1. Notation

As we have mentioned in the Notation Section at the beginning of this thesis, we rescale the Lebesgue measure by a factor $1/\epsilon$ since the thin domain R^ϵ degenerates into a line segment. Thus, we deal with the singular measure $\rho_\epsilon(\mathcal{O}) = \epsilon^{-1} |\mathcal{O}|$ in order to preserve the relative capacity of a mensurable subset $\mathcal{O} \subset R^\epsilon$. Then, we consider the following inner products

$$(u, v)_\epsilon = \epsilon^{-1} \int_{R^\epsilon} u \, v \, dx dy, \quad \forall\, u, v \in L^2(R^\epsilon),$$

and

$$a_\epsilon(u, v) = \epsilon^{-1} \int_{R^\epsilon} \{ \nabla u \cdot \nabla v + u \, v \} dx dy, \quad \forall\, u, v \in H^1(R^\epsilon),$$

which induce the norms

$$|||\varphi|||_{L^2(R^\epsilon)} = \epsilon^{-1/p}||\varphi||_{L^2(R^\epsilon)}, \quad \forall\varphi \in L^2(R^\epsilon),$$
$$|||\varphi|||_{H^1(R^\epsilon)} = \epsilon^{-1/p}||\varphi||_{H^1(R^\epsilon)}, \quad \forall\varphi \in H^1(R^\epsilon).$$

Consequently, with these norms we introduce the Lebesgue and Sobolev spaces $L^2(R^\epsilon;\rho_\epsilon)$ and $H^1(R^\epsilon;\rho_\epsilon)$.

Therefore, the variational formulation of (3.0.1) is equivalent to find $w^\epsilon \in H^1(R^\epsilon;\rho^\epsilon)$ such that

$$a_\epsilon(\varphi, w^\epsilon) = (\varphi, f^\epsilon)_\epsilon, \quad \forall\,\varphi \in H^1(R^\epsilon;\rho_\epsilon). \tag{3.1.75}$$

Furthermore, we consider the sesquilinear a_0 in $H^1(0,1)\times H^1(0,1)$ and the inner product $(\cdot,\cdot)_0$ given by

$$a_0(u,v) = \int_0^1 \left\{\hat{q}\,u_x v_x + \Big(\frac{|Y^*|}{L_1} + p\Big)\,uv\right\} dx, \quad \forall\,u,v \in H^1(0,1), \tag{3.1.76}$$

$$(u,v)_0 = \int_0^1 \Big(\frac{|Y^*|}{L_1} + p\Big)\,uv\,dx, \quad \forall\,u,v \in L^2(0,1), \tag{3.1.77}$$

where $\hat{q}$ and p are defined in (3.1.18). Then, the variational formulation of the limit problem (3.1.17) is: find $u_0 \in H^1(0,1)$ such that

$$a_0(\phi, u_0) = (\phi, f_0)_0,\ \forall\phi \in H^1(0,1), \tag{3.1.78}$$

with $f_0(x) = \dfrac{\hat{f}(x)}{\frac{|Y^*|}{L_1} + p}$.

3.1.3.2. First-order corrector

We construct now the first-order corrector $\kappa^\epsilon \in H^1(R^\epsilon)$ which allows us to get strong convergence in H^1-norm of the sequence of solutions $\{w^\epsilon\}$.

Since the oscillations at the lower and the upper boundary have different order there is a natural division of the problem into two parts. As we made for the oscillating domain Ω^ϵ we divide the thin domain R^ϵ in two parts

$$R^\epsilon_+ = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1,\ -\epsilon h_0 < y < \epsilon g(x/\epsilon)\},$$
$$R^\epsilon_- = \{(x,y)) \in \mathbb{R}^2 : 0 < x < 1,\ -\epsilon h(x/\epsilon^\alpha) < y < -\epsilon h_0\}.$$

Observe that $R^\epsilon = \text{Int}\ (\overline{R^\epsilon_+ \cup R^\epsilon_-})$.

We start defining the corrector function in R^ϵ_+. To do so, according to Bensoussan, Lions and Papanicolaou in [19] we use the auxiliary function $X = X(y_1,y_2)$ originally defined in the reference cell Y^* by the problem (3.1.20). Observe that by the L_1-periodicity of X and the relation between the basic cell Y^*, see (3.1.19), and the thin domain R^ϵ_+ we may consider

$$X^\epsilon(x,y) = X\Big(\frac{x}{\epsilon}, \frac{y}{\epsilon}\Big), \quad \forall(x,y) \in R^\epsilon_+, \tag{3.1.79}$$

which is a well-defined function in $H^1(R_+^\epsilon)$. Furthermore, under these considerations, we can obtain some estimates for X^ϵ and its derivatives on R_+^ϵ. It is easy to see that

$$\begin{aligned}\|X^\epsilon\|^2_{L^2(R_+^\epsilon)} &= \int_{R_+^\epsilon} |X(x/\epsilon, y/\epsilon)|^2\, dxdy \\ &= \epsilon^2 \int_0^{1/\epsilon} \int_{-h_0}^{g(x)} |X(x,y)|^2\, dydx \\ &\leq C \sum_{k=1}^{1/\epsilon L_1} \epsilon^2 \int_{Y^*} |X(y_1,y_2)|^2\, dy_1 dy_2 \\ &\leq \epsilon C \|X\|^2_{L^2(Y^*)},\end{aligned} \tag{3.1.80}$$

for some positive constant C independent of ϵ.

We also consider the following functions

$$X_1^\epsilon(x,y) = \frac{\partial X}{\partial y_1}\big(\frac{x}{\epsilon}, \frac{y}{\epsilon}\big) \text{ and } X_2^\epsilon(x,y) = \frac{\partial X}{\partial y_2}\big(\frac{x}{\epsilon}, \frac{y}{\epsilon}\big),\ \forall (x,y) \in R_+^\epsilon, \tag{3.1.81}$$

and with similar computations as in (3.1.80) we have the following estimates for their norms on R_+^ϵ

$$\|X_1^\epsilon\|^2_{L^2(R_+^\epsilon)} \leq \epsilon C \Big\|\frac{\partial X}{\partial y_1}\Big\|^2_{L^2(Y^*)}, \tag{3.1.82}$$

$$\|X_2^\epsilon\|^2_{L^2(R_+^\epsilon)} \leq \epsilon C \Big\|\frac{\partial X}{\partial y_2}\Big\|^2_{L^2(Y^*)}. \tag{3.1.83}$$

Then, in the upper part R_+^ϵ we define the corrector as

$$\kappa^\epsilon(x,y) := -\epsilon X\Big(\frac{x}{\epsilon}, \frac{y}{\epsilon}\Big)\frac{\partial u_0}{\partial x}(x), \quad \forall (x,y) \in R_+^\epsilon, \tag{3.1.84}$$

where X is the solution of (3.1.20) and $u_0 \in H^2(0,1) \cap C^1(0,1)$ is the solution of the limit problem (3.1.17).

Remark 3.1.11. *Observe that in case $h(\cdot) \equiv 0$ the first-order corrector was obtained in [96] and, as we might expect, the definition coincides with* (3.1.84).

We define now the appropriate corrector term in the highly oscillating part R_-^ϵ. Actually, this is the main contribution of this section.

We begin constructing a family of functions $\{v^\epsilon\}_{\epsilon>0}$ using the same kind of problems defined in rectangles as in paragraphs **d)** and **f.3)**. Then, we consider the following family of boundary value problems

$$\begin{cases} \dfrac{\partial^2 v_n^\epsilon}{\partial x_1^2} + \dfrac{1}{\epsilon^2}\dfrac{\partial^2 v_n^\epsilon}{\partial x_2^2} = 0 \quad \text{in } Q_n^\epsilon, \\ \dfrac{\partial v_n^\epsilon}{\partial \nu} = 0 \quad \text{on } \partial Q_n^\epsilon \backslash \Gamma_n^\epsilon, \\ v_n^\epsilon(x_1,x_2) = u_0 - \epsilon X\Big(\dfrac{x_1}{\epsilon}, -h_0\Big)\dfrac{\partial u_0}{\partial x_1} \quad \text{on } \Gamma_n^\epsilon, \end{cases} \tag{3.1.85}$$

where X is the L_1-periodic solution of problem (3.1.20), Γ_n^ϵ is the top of the rectangle Q_n^ϵ which, as in (3.1.34), is given by

$$Q_n^\epsilon = \{(x_1, x_2) \,|\, \gamma_{n,\epsilon} < x_1 < \gamma_{n+1,\epsilon},\ -h_1 < x_2 < -h_0\}, \quad n = 0, 1, \dots, N_\epsilon.$$

Recall that N_ϵ denotes the largest integer such that $N_\epsilon L_2 \epsilon^\alpha < 1$ and $\gamma_{n,\epsilon}$ are the points defined in item **(d)** above, see also Figure 3.4. Recall that they satisfy $h(\frac{\gamma_{n,\epsilon}}{\epsilon^\alpha}) = h_0$.

Since the solution of the auxiliary problem (3.1.20) satisfies $X \in H^2(Y^*)$ and $u_0 \in H^2(0,1) \cap C^1(0,1)$ we can ensure that

$$u_0 - \epsilon X\Big(\frac{\cdot}{\epsilon}, -h_0\Big)\frac{\partial u_0}{\partial x_1} \in H^1(\gamma_{n,\epsilon}, \gamma_{n+1,\epsilon})$$

for all $n \in \{0, \cdots, N_\epsilon\}$. Therefore, recalling definition (3.1.81) we have the following estimate from Lemma 3.1.4

$$\begin{aligned} \Big\|\frac{\partial v_n^\epsilon}{\partial x_1}\Big\|^2_{L^2(Q_n^\epsilon)} + \frac{1}{\epsilon^2}\Big\|\frac{\partial v_n^\epsilon}{\partial x_2}\Big\|^2_{L^2(Q_n^\epsilon)} \le C\epsilon^{\alpha-1}\Bigg\{&\Big\|\frac{\partial u_0}{\partial x_1} - X_1^\epsilon(\cdot, -\epsilon h_0)\frac{\partial u_0}{\partial x_1}\Big\|^2_{L^2(\gamma_{n,\epsilon},\gamma_{n+1,\epsilon})} \\ &+ \Big\|\epsilon X\Big(\frac{\cdot}{\epsilon}, -h_0\Big)\frac{\partial^2 u_0}{\partial x_1^2}\Big\|^2_{L^2(\gamma_{n,\epsilon},\gamma_{n+1,\epsilon})}\Bigg\}. \end{aligned} \tag{3.1.86}$$

Hence, we define the function $v^\epsilon \in H^1(R_-^\epsilon)$ as follows

$$v^\epsilon(x,y) = v_n^\epsilon(x, y/\epsilon), \qquad \text{as } (x, y/\epsilon) \in Q_n^\epsilon.$$

Performing the change of variables $x = x_1$ and $y = \epsilon x_2$ and using (3.1.85) we have that v^ϵ satisfies the following inequality

$$\begin{aligned} &\frac{1}{\epsilon}\Big\|\frac{\partial v^\epsilon}{\partial x}\Big\|^2_{L^2(R_-^\epsilon)} + \frac{1}{\epsilon}\Big\|\frac{\partial v^\epsilon}{\partial y}\Big\|^2_{L^2(R_-^\epsilon)} \le \sum_{n=1}^{N_\epsilon}\Bigg(\Big\|\frac{\partial v_n^\epsilon}{\partial x_1}\Big\|^2_{L^2(Q_n^\epsilon)} + \frac{1}{\epsilon^2}\Big\|\frac{\partial v_n^\epsilon}{\partial x_2}\Big\|^2_{L^2(Q_n^\epsilon)}\Bigg) \\ &\le \sum_{n=1}^{N_\epsilon} C\,\epsilon^{\alpha-1}\Big\|\frac{\partial u_0}{\partial x_1} - X_1^\epsilon(\cdot, -\epsilon h_0)\frac{\partial u_0}{\partial x_1}\Big\|^2_{L^2(\gamma_{n,\epsilon},\gamma_{n+1,\epsilon})} \\ &+ \sum_{n=1}^{N_\epsilon} C\,\epsilon^{\alpha-1}\Big\|\epsilon X\Big(\frac{\cdot}{\epsilon}, -h_0\Big)\frac{\partial^2 u_0}{\partial x_1^2}\Big\|^2_{L^2(\gamma_{n,\epsilon},\gamma_{n+1,\epsilon})} \\ &\le C\,\epsilon^{\alpha-1}\|1 - X_1^\epsilon(\cdot, -\epsilon h_0)\|^2_{L^2(0,1)} + C\,\epsilon^\alpha\Big\|\frac{\partial^2 u_0}{\partial x_1^2}\Big\|^2_{L^2(0,1)} \\ &\le C\,\epsilon^{\alpha-1}. \end{aligned} \tag{3.1.87}$$

Now we are in conditions to define the appropriate first-order corrector in R^ϵ

$$\kappa^\epsilon(x,y) = \begin{cases} -\epsilon X^\epsilon(x,y)\frac{\partial u_0}{\partial x}(x), & \text{if } (x,y) \in R_+^\epsilon, \\ -u_0(x) + v^\epsilon(x,y), & \text{if } (x,y) \in R_-^\epsilon. \end{cases} \tag{3.1.88}$$

Recall that $X^\epsilon(x,y) = X\Big(\frac{x}{\epsilon},\frac{y}{\epsilon}\Big) \quad \forall (x,y) \in R^\epsilon$.

Let us point out that κ^ϵ belongs to $H^1(R^\epsilon)$. Indeed, observe that on one hand, $\kappa^\epsilon_{|R^\epsilon_+} \in H^1(R^\epsilon_+)$ and $\kappa^\epsilon_{|R^\epsilon_-} \in H^1(R^\epsilon_-)$ since by the regularity of the weak solutions we can ensure that $u_0 \in H^2(0,1) \cap C^1(0,1)$, $X \in H^2(Y^*)$ and $v^\epsilon(x,y) \in H^1(R^\epsilon)$. On the other hand, the traces of $\kappa^\epsilon_{|R^\epsilon_+}$ and $\kappa^\epsilon_{|R^\epsilon_-}$ coincide on $\{y = -\epsilon h_0\}$.

To conclude this subsection we prove a necessary convergence for the corrector function κ^ϵ.

Proposition 3.1.12. *Let κ^ϵ the corrector function defined in* (3.1.88). *Then,*

$$|||\kappa^\epsilon|||_{L^2(R^\epsilon)} \equiv \epsilon^{-1/2}||\kappa^\epsilon||_{L^2(R^\epsilon)} \xrightarrow{\epsilon\to 0} 0. \tag{3.1.89}$$

Proof. From definition (3.1.88) we have

$$|||\kappa^\epsilon|||_{L^2(R^\epsilon)} = \Big|\Big|\Big|\epsilon X^\epsilon \frac{\partial u_0}{\partial x}\Big|\Big|\Big|_{L^2(R^\epsilon_+)} + \Big|\Big|\Big| - u_0 + v^\epsilon \Big|\Big|\Big|_{L^2(R^\epsilon_-)}. \tag{3.1.90}$$

On one hand, in view of (3.1.80) and since $u_0 \in C^1(0,1)$ we immediately obtain

$$\Big|\Big|\Big|\epsilon X^\epsilon \frac{\partial u_0}{\partial x}\Big|\Big|\Big|_{L^2(R^\epsilon_+)} \leq \epsilon \Big|\Big|\frac{\partial u_0}{\partial x}\Big|\Big|_{L^\infty(0,1)} |||X^\epsilon|||_{L^2(R^\epsilon_+)} \leq C\epsilon.$$

Consequently,

$$\Big|\Big|\Big|\epsilon X^\epsilon \frac{\partial u_0}{\partial x}\Big|\Big|\Big|_{L^2(R^\epsilon_+)} \xrightarrow{\epsilon\to 0} 0. \tag{3.1.91}$$

On the other hand, since v^ϵ satisfies (3.1.85) we can write

$$\begin{aligned} -u_0 + v^\epsilon(x,y) &= -v^\epsilon(x,-\epsilon h_0) - \epsilon X\Big(\frac{x}{\epsilon}, -h_0\Big)\frac{\partial u_0}{\partial x} + v^\epsilon(x,y) \\ &= -\epsilon X\Big(\frac{x}{\epsilon}, -h_0\Big)\frac{\partial u_0}{\partial x} + \int_y^{-\epsilon h_0} -\frac{\partial v^\epsilon}{\partial y}(x,s)\,ds, \quad \forall (x,y) \in R^\epsilon_-. \end{aligned}$$

Hence, using estimates (3.1.87) and taking into account that $X \in H^2(Y^*)$ and $u_0 \in C^1(0,1)$ we get

$$|||-u_0 + v^\epsilon|||_{L^2(R^\epsilon_-)} \leq C\Big|\Big|\Big|\epsilon X\Big(\frac{\cdot}{\epsilon}, -h_0\Big)\Big|\Big|\Big|_{L^2(R^\epsilon_-)} + C\Big|\Big|\Big|\frac{\partial v^\epsilon}{\partial y}\Big|\Big|\Big|_{L^2(R^\epsilon_-)} \xrightarrow{\epsilon\to 0} 0. \tag{3.1.92}$$

Observe that $\big|\big|\big|\epsilon X\big(\frac{\cdot}{\epsilon}, -h_0\big)\big|\big|\big|_{L^2(R^\epsilon_-)}$ tends to zero since the following inequality holds

$$\begin{aligned} \Big|\Big|\Big|\epsilon X\Big(\frac{\cdot}{\epsilon}, -h_0\Big)\Big|\Big|\Big|^2_{L^2(R^\epsilon_-)} &\leq \frac{1}{\epsilon}\int_{R^\epsilon_-} \Big|\epsilon X\Big(\frac{x}{\epsilon}, -h_0\Big)\Big|^2 dydx \\ &\leq ||X||^2_{L^\infty(Y^*)} \int_{R^\epsilon_-} \epsilon\, dydx \\ &\leq \epsilon^2(h_1 - h_0)||X||^2_{L^\infty(Y^*)}. \end{aligned}$$

Therefore, in view of (3.1.90) and convergences (3.1.91) and (3.1.92) we obtain

$$|||\kappa^\epsilon|||_{L^2(R^\epsilon)} \xrightarrow{\epsilon\to 0} 0$$

completing the proof. □

3.1.3.3. Corrector result

In this subsection we show the corrector result, which allows us to improve the convergences obtained in Theorem 3.1.5. Then, we will prove that using the corrector function κ^ϵ defined in (3.1.88) we get a strong convergence in H^1-norm.

Theorem 3.1.13. *Let u^ϵ be the solution of problem* (3.0.1) *with $f^\epsilon \in L^2(R^\epsilon)$ satisfying*

$$|||f^\epsilon|||_{L^2(R^\epsilon)} \leq C,$$

for some $C > 0$ independent of ϵ. Consider the family of functions $\hat{f}^\epsilon \in L^2(0,1)$ defined by

$$\hat{f}^\epsilon(x) = \epsilon^{-1} \int_{-\epsilon h(x/\epsilon^\alpha)}^{\epsilon g(x/\epsilon)} f^\epsilon(x,y)\, dy. \tag{3.1.93}$$

If $\hat{f}^\epsilon \overset{\epsilon\to 0}{\rightharpoonup} \hat{f}$ in $L^2(0,1)$, then

$$\lim_{\epsilon\to 0} |||w^\epsilon - u_0 - \kappa^\epsilon|||_{H^1(R^\epsilon)} = 0, \tag{3.1.94}$$

where κ^ϵ is the first-order corrector defined in (3.1.88)*, and $u_0 \in H^2(0,1) \cap C^1(0,1)$ is the unique solution of the homogenized equation* (3.1.17).

Proof. From the variational formulation of (3.0.1), see (3.1.75), we have

$$a_\epsilon(\varphi, w^\epsilon) = (\varphi, f^\epsilon)_\epsilon, \quad \forall\, \varphi \in H^1(R^\epsilon).$$

Thus, taking $u_0 + \kappa^\epsilon \in H^1(R^\epsilon)$ as a test function, we obtain

$$\begin{aligned} |||w^\epsilon - u_0 - \kappa^\epsilon|||^2_{H^1(R^\epsilon)} &= a_\epsilon(w^\epsilon - u_0 - \kappa^\epsilon, w^\epsilon - u_0 - \kappa^\epsilon) \\ &= a_\epsilon(w^\epsilon, w^\epsilon - u_0 - \kappa^\epsilon) - a_\epsilon(u_0 + \kappa^\epsilon, w^\epsilon) + a_\epsilon(u_0 + \kappa^\epsilon, u_0 + \kappa^\epsilon) \\ &= (w^\epsilon - 2(u_0 + \kappa^\epsilon), f^\epsilon)_\epsilon + a_\epsilon(u_0 + \kappa^\epsilon, u_0 + \kappa^\epsilon). \end{aligned} \tag{3.1.95}$$

From convergence (3.1.16) and using the change of variables $(x, y) \to (x, y/\epsilon)$ it is easy to check that

$$|||w^\epsilon - u_0|||_{L^2(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0.$$

Therefore, since $|||f^\epsilon|||_{L^2(R^\epsilon)} \leq C$ we get

$$\left|(w^\epsilon - u_0, f^\epsilon)_\epsilon\right| \leq |||w^\epsilon - u_0|||_{L^2(R^\epsilon)} |||f^\epsilon|||_{L^2(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0. \tag{3.1.96}$$

Moreover, by Proposition 3.1.12 we obtain

$$\begin{aligned} \left|(\kappa^\epsilon, f^\epsilon)_\epsilon\right| &\leq \epsilon^{-1} \|\kappa^\epsilon\|_{L^2(R^\epsilon)} \|f^\epsilon\|_{L^2(R^\epsilon)} \\ &= \epsilon^{-1/2} \|\kappa^\epsilon\|_{L^2(R^\epsilon)} \epsilon^{-1/2} \|f^\epsilon\|_{L^2(R^\epsilon)} \\ &= |||\kappa^\epsilon|||_{L^2(R^\epsilon)} |||f^\epsilon|||_{L^2(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0. \end{aligned} \tag{3.1.97}$$

Since $\hat{f}^\epsilon \stackrel{\epsilon\to 0}{\rightharpoonup} \hat{f}$ in $L^2(0,1)$, we obtain that

$$\begin{aligned}
(u_0, f^\epsilon)_\epsilon &= \epsilon^{-1}\int_0^1 u_0(x)\int_{-\epsilon h(x/\epsilon^\alpha)}^{\epsilon g(x/\epsilon)} f^\epsilon(x,y)\,dydx \\
&= \int_0^1 u_0(x)\hat{f}^\epsilon(x)\,dx \\
&\stackrel{\epsilon\to 0}{\longrightarrow} \int_0^1 \Big(\frac{|Y^*|}{L_1}+p\Big)u_0(x)f_0(x)\,dx, \qquad (3.1.98)
\end{aligned}$$

where $f_0 := \dfrac{\hat{f}(x)}{\frac{|Y^*|}{L_1}+p}$.

Therefore, we get from (3.1.96), (3.1.97), (3.1.98) and (3.1.77) that

$$(w^\epsilon - 2(u_0+\kappa^\epsilon), f^\epsilon)_\epsilon = (w^\epsilon - u_0, f^\epsilon)_\epsilon - (u_0, f^\epsilon)_\epsilon - 2(\kappa^\epsilon, f^\epsilon)_\epsilon \stackrel{\epsilon\to 0}{\longrightarrow} -(u_0, f_0)_0. \quad (3.1.99)$$

Now we show that

$$a_\epsilon(u_0+\kappa^\epsilon, u_0+\kappa^\epsilon) \stackrel{\epsilon\to 0}{\longrightarrow} a_0(u_0,u_0). \qquad (3.1.100)$$

First of all, observe that in view of definitions (3.1.88), (3.1.79) and (3.1.81) we have

$$\begin{aligned}
a_\epsilon(u_0+\kappa^\epsilon, u_0+\kappa^\epsilon) &= \epsilon^{-1}\int_{R^\epsilon}\big\{|\nabla(u_0+\kappa^\epsilon)|^2 + |u_0+\kappa^\epsilon|^2\big\}dxdy \\
&= \epsilon^{-1}\int_{R^\epsilon}|u_0+\kappa^\epsilon|^2\,dxdy \\
&+ \epsilon^{-1}\int_{R^\epsilon_+}\Big\{\Big(\frac{\partial u_0}{\partial x}\Big)^2\Big((1-X_1^\epsilon)^2+(X_2^\epsilon)^2\Big)\Big\}dxdy \\
&- 2\int_{R^\epsilon_+}(1-X_1^\epsilon)\frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2}X^\epsilon dxdy + \epsilon\int_{R^\epsilon_+}(X^\epsilon)^2\Big(\frac{\partial^2 u_0}{\partial x^2}\Big)^2 dxdy \\
&+ \epsilon^{-1}\int_{R^\epsilon_-}\Big\{\Big(\frac{\partial v^\epsilon}{\partial x}\Big)^2+\Big(\frac{\partial v^\epsilon}{\partial y}\Big)^2\Big\}dxdy.
\end{aligned}$$

Now, we pass to the limit on every term on the right-hand side

- $\epsilon^{-1}\displaystyle\int_{R^\epsilon}|u_0+\kappa^\epsilon|^2\,dxdy \stackrel{\epsilon\to 0}{\longrightarrow} \int_0^1\Big(\frac{|Y^*|}{L_1}+p\Big)u_0^2\,dx.$

  In order to obtain that, we need to pass to the limit on the following expression

$$\epsilon^{-1}\int_{R^\epsilon}|u_0+\kappa^\epsilon|^2\,dxdy = \epsilon^{-1}\int_{R^\epsilon}\big\{u_0^2+2u_0\kappa^\epsilon+(\kappa^\epsilon)^2\big\}\,dxdy. \qquad (3.1.101)$$

  Taking into account that $u_0 \in C^1(0,1)$ and using Proposition 3.1.12 we get

$$\epsilon^{-1}\int_{R^\epsilon}\big\{2u_0\kappa^\epsilon+(\kappa^\epsilon)^2\big\}\,dxdy \le C\epsilon^{-1/2}\|\kappa^\epsilon\|_{L^2(R^\epsilon)} + \epsilon^{-1}\|\kappa^\epsilon\|^2_{L^2(R^\epsilon)} \stackrel{\epsilon\to 0}{\longrightarrow} 0. \qquad (3.1.102)$$

Moreover, from (3.1.5) and (3.1.6) we have that

$$\epsilon^{-1} \int_{R^\epsilon} (u_0)^2 dxdy \stackrel{\epsilon\to 0}{\longrightarrow} \int_0^1 \Big(\frac{|Y^*|}{L_1} + p\Big) u_0^2 \, dx. \tag{3.1.103}$$

Hence, from (3.1.101)-(3.1.103) we obtain the desired convergence.

- $\epsilon^{-1} \int_{R_+^\epsilon} \Big\{ \Big(\frac{\partial u_0}{\partial x}\Big)^2 \Big((1 - X_1^\epsilon)^2 + (X_2^\epsilon)^2 \Big) \Big\} dxdy \stackrel{\epsilon\to 0}{\longrightarrow} \int_0^1 \hat{q} \Big(\frac{\partial u_0}{\partial x}\Big)^2 dx,$
  where $\hat{q} = \frac{1}{L_1} \int_{Y^*} \Big\{ 1 - \frac{\partial X}{\partial y_1}(y_1, y_2) \Big\} dy_1 dy_2$.

In view of definitions (3.1.79) and (3.1.81) one gets

$$\begin{aligned} &\epsilon^{-1} \int_{R_+^\epsilon} \Big\{ \Big(\frac{\partial u_0}{\partial x}\Big)^2 \Big((1 - X_1^\epsilon)^2 + (X_2^\epsilon)^2 \Big) \Big\} dxdy \\ &\qquad = \int_0^1 \Big(\frac{\partial u_0}{\partial x}\Big)^2 \int_{-h_0}^{g(x/\epsilon)} \Bigg(\Big(1 - \frac{\partial X}{\partial y_1}(\frac{x}{\epsilon}, z)\Big)^2 + \Big(\frac{\partial X}{\partial y_2}(\frac{x}{\epsilon}, z)\Big)^2 \Bigg) dzdx. \end{aligned}$$

Hence, taking into account that

$$\int_{-h_0}^{g(y_1)} \Bigg(\Big(1 - \frac{\partial X}{\partial y_1}(y_1, z)\Big)^2 + \Big(\frac{\partial X}{\partial y_2}(y_1, z)\Big)^2 \Bigg) dz$$

is a L_1-periodic function we obtain by the Average Convergence for Periodic Functions (see, e.g., [52, p. xvi])

$$\begin{aligned} &\epsilon^{-1} \int_{R_+^\epsilon} \Big\{ \Big(\frac{\partial u_0}{\partial x}\Big)^2 \Big((1 - X_1^\epsilon)^2 + (X_2^\epsilon)^2 \Big) \Big\} dxdy \\ &\qquad \stackrel{\epsilon\to 0}{\longrightarrow} \int_0^1 \Big(\frac{\partial u_0}{\partial x}\Big)^2 \frac{1}{L_1} \int_{Y^*} \Bigg(\Big(1 - \frac{\partial X}{\partial y_1}\Big)^2 + \Big(\frac{\partial X}{\partial y_2}\Big)^2 \Bigg) dy_1 dy_2 dx. \end{aligned} \tag{3.1.104}$$

Now, since X satisfies (3.1.20), we have that

$$\int_{Y^*} |\nabla_{y_1,y_2} X|^2 dy_1 dy_2 = \int_{B_1} N_1 \, X \, dS$$

where $N = (N_1, N_2)$ is the unit outward normal to B_1, the upper boundary of ∂Y^*. Hence, we obtain from

$$\int_{B_1} N_1 \, X \, dS = \int_{Y^*} \mathrm{div}_{y_1,y_2} \begin{pmatrix} X \\ 0 \end{pmatrix} dy_1 dy_2 = \int_{Y^*} \frac{\partial X}{\partial y_1} \, dy_1 dy_2$$

that

$$\int_{Y^*} |\nabla_{y_1,y_2} X|^2 dy_1 dy_2 = \int_{Y^*} \frac{\partial X}{\partial y_1} \, dy_1 dy_2. \tag{3.1.105}$$

Thus, due to (3.1.105), we have

$$
\begin{aligned}
&\int_{Y^*}\left(\left(1-\frac{\partial X}{\partial y_1}\right)^2+\left(\frac{\partial X}{\partial y_2}\right)^2\right)dy_1dy_2\\
&=\int_{Y^*}\left\{1-2\,\frac{\partial X}{\partial y_1}+|\nabla_{y_1,y_2}X|^2\right\}dy_1dy_2\\
&=\int_{Y^*}\left\{1-\frac{\partial X}{\partial y_1}\right\}dy_1dy_2, \qquad (3.1.106)
\end{aligned}
$$

Consequently, from (3.1.104) and (3.1.106) we get the requested convergence

$$
\begin{aligned}
\epsilon^{-1}&\int_{R_+^\epsilon}\left\{\left(\frac{\partial u_0}{\partial x}\right)^2\left((1-X_1^\epsilon)^2+(X_2^\epsilon)^2\right)\right\}dxdy\\
&\overset{\epsilon\to 0}{\longrightarrow}\int_0^1\left(\frac{\partial u_0}{\partial x}\right)^2\frac{1}{L_1}\int_{Y^*}\left\{1-\frac{\partial X}{\partial y_1}\right\}dy_1dy_2dx.
\end{aligned}
$$

- $-2\displaystyle\int_{R_+^\epsilon}(1-X_1^\epsilon)\frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2}X^\epsilon dxdy+\epsilon\int_{R_+^\epsilon}(X^\epsilon)^2\left(\frac{\partial^2 u_0}{\partial x^2}\right)^2dxdy\overset{\epsilon\to 0}{\longrightarrow}0.$

  On one hand, taking into account the estimate (3.1.82), $X\in H^2(Y^*)$ and $u_0\in H^2(0,1)\cap C^1(0,1)$ by Holder's inequality we have

$$
\begin{aligned}
\Big|-2\int_{R_+^\epsilon}(1-X_1^\epsilon)\frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2}X^\epsilon dxdy\Big| &\le \Big|\Big|(1-X_1^\epsilon)\frac{\partial u_0}{\partial x}\Big|\Big|_{L^2(R_+^\epsilon)}\Big|\Big|\frac{\partial^2 u_0}{\partial x^2}X^\epsilon\Big|\Big|_{L^2(R_+^\epsilon)}\\
&\le \Big|\Big|\frac{\partial u_0}{\partial x}\Big|\Big|_{L^\infty(0,1)}||(1-X_1^\epsilon)||_{L^2(R_+^\epsilon)}||X||_{L^\infty(Y^*)}\Big|\Big|\frac{\partial^2 u_0}{\partial x^2}\Big|\Big|_{L^2(R_+^\epsilon)}\\
&\le C\epsilon.
\end{aligned}
$$

  On the other hand,

$$
\Big|\epsilon\int_{R_+^\epsilon}(X^\epsilon)^2\left(\frac{\partial^2 u_0}{\partial x^2}\right)^2dxdy\Big|\le\epsilon||X||^2_{L^\infty(Y^*)}\Big|\Big|\frac{\partial^2 u_0}{\partial x^2}\Big|\Big|^2_{L^2(R_+^\epsilon)}\le C\epsilon^2.
$$

- $\epsilon^{-1}\displaystyle\int_{R_-^\epsilon}\left\{\left(\frac{\partial v^\epsilon}{\partial x}\right)^2+\left(\frac{\partial v^\epsilon}{\partial y}\right)^2\right\}dxdy\overset{\epsilon\to 0}{\longrightarrow}0.$

  It is an immediately consequence of estimate (3.1.87).

Hence, thanks to convergences above we obtain

$$
a_\epsilon(u_0+\kappa^\epsilon,u_0+\kappa^\epsilon)\to\int_0^1\left(\frac{|Y^*|}{L_1}+p\right)u_0^2\,dx+\int_0^1\hat q\left(\frac{\partial u_0}{\partial x}\right)^2dx,
$$

which proves (3.1.100).

Finally, in accordance with (3.1.95) and taking into account convergences (3.1.99), (3.1.100) and using that u_0 satisfies (3.1.17) we complete the proof getting

$$
|||w^\epsilon-u_0-\kappa^\epsilon|||^2_{H^1(R^\epsilon)}\overset{\epsilon\to 0}{\longrightarrow}a_0(u_0,u_0)-(u_0,f_0)_0=0.
$$

□

As a consequence of Theorem 3.1.13 we can assert that the roughness is so strong at the bottom boundary that w^ϵ restricted to the extremely oscillating part, R^ϵ_-, tends to be constant. Indeed, in the next corollary we prove that the partial derivatives of w^ϵ restricted to R^ϵ_- tends to zero.

Corollary 3.1.14. *Assume hypothesis of Theorem 3.1.13. Then*

$$\left|\left|\left|\frac{\partial w^\epsilon}{\partial x}\right|\right|\right|_{L^2(R^\epsilon_-)} + \left|\left|\left|\frac{\partial w^\epsilon}{\partial y}\right|\right|\right|_{L^2(R^\epsilon_-)} \stackrel{\epsilon\to 0}{\longrightarrow} 0.$$

Proof. From Minkowski's inequality and definition (3.1.88) we have

$$\begin{aligned}\left|\left|\left|\frac{\partial w^\epsilon}{\partial x}\right|\right|\right|_{L^2(R^\epsilon_-)} + \left|\left|\left|\frac{\partial w^\epsilon}{\partial y}\right|\right|\right|_{L^2(R^\epsilon_-)} &\le \left|\left|\left|\frac{\partial w^\epsilon}{\partial x} - \frac{\partial u_0}{\partial x} - \frac{\partial \kappa^\epsilon}{\partial x}\right|\right|\right|_{L^2(R^\epsilon_-)} \\ + \left|\left|\left|\frac{\partial w^\epsilon}{\partial y} - \frac{\partial \kappa^\epsilon}{\partial y}\right|\right|\right|_{L^2(R^\epsilon_-)} &+ \left|\left|\left|\frac{\partial u_0}{\partial x} + \frac{\partial \kappa^\epsilon}{\partial x}\right|\right|\right|_{L^2(R^\epsilon_-)} + \left|\left|\left|\frac{\partial \kappa^\epsilon}{\partial y}\right|\right|\right|_{L^2(R^\epsilon_-)} \\ \le |||w^\epsilon - u_0 - \kappa^\epsilon|||_{H^1(R^\epsilon_-)} &+ \left|\left|\left|\frac{\partial v^\epsilon}{\partial x}\right|\right|\right|_{L^2(R^\epsilon_-)} + \left|\left|\left|\frac{\partial v^\epsilon}{\partial y}\right|\right|\right|_{L^2(R^\epsilon_-)}.\end{aligned}$$

Therefore, using (3.1.94) and (3.1.87) we get the desired convergence. □

To conclude this section we would like to point out that the results introduced in this section allows us also to obtain the first order corrector for the particular case where the thin domain presents fast oscillations only at one of the two boundaries.

Thus, using the same notation as in definition (3.1.88) , if the upper boundary does not oscillate, say the function g is the constant function $g(x) \equiv g_0 > 0$, then, since the function X is constant, it is easy to see from the results of this section that the first corrector function is given by

$$\kappa^\epsilon(x,y) = \begin{cases} 0, \text{ if } (x,y) \in R^\epsilon_+, \\ -u_0 + \widetilde{v}^\epsilon(x,y), \text{ if } (x,y) \in R^\epsilon_-, \end{cases}$$

where $\widetilde{v}^\epsilon(x,y) = \widetilde{v}^\epsilon_n(x, y/\epsilon)$, as $(x, y/\epsilon) \in Q^\epsilon_n$ and

$$\begin{cases} \dfrac{\partial^2 \widetilde{v}^\epsilon_n}{\partial x_1^2} + \dfrac{1}{\epsilon^2}\dfrac{\partial^2 \widetilde{v}^\epsilon_n}{\partial x_2^2} = 0 \quad \text{in } Q^\epsilon_n, \\ \dfrac{\partial \widetilde{v}^\epsilon_n}{\partial \nu} = 0 \quad \text{on } \partial Q^\epsilon_n \backslash \Gamma^\epsilon_n, \\ \widetilde{v}^\epsilon_n(x_1, x_2) = u_0 \quad \text{on } \Gamma^\epsilon_n. \end{cases}$$

3.1.4. Perforated thin domain with doubly oscillatory boundary

In this section we explain through an example how the method introduced in previous subsections can be applied to a thin domain with doubly oscillatory boundary which also presents holes. Thus, we study the behavior of the solutions of the Neumann problem (3.0.1) where R^ϵ is a thin perforated domain which presents oscillations at the top and the bottom boundary in the same way as the previous sections.

First of all, we describe in detail the model perforated domain that we will consider in this section.

Using the same notation as Subsection 3.1.1, we consider

$$R^\epsilon = \Big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ -\epsilon\, h\Big(\frac{x}{\epsilon^\alpha}\Big) < y < \epsilon\, g\Big(\frac{x}{\epsilon}\Big)\Big\}, \quad \text{with } \alpha > 1,$$
$$Y^* = \{(y_1, y_2) \in \mathbb{R}^2 \mid 0 < y_1 < L_1,\ -h_0 < y_2 < g(y_1)\}.$$

Then, we perforate R^ϵ by removing from it a set T^ϵ of periodically distributed holes defined as follows. Let T be an open subset of Y^* with a smooth boundary ∂T and such that $\overline{T} \subset Y^*$, that is ∂T does not intersect ∂Y^*. Then, we consider the perforated representative cell given by

$$Y_p^* = Y^* \setminus \overline{T}, \tag{3.1.107}$$

which we assume that it is a connected domain.

Figure 3.5: An example of perforated cell Y_p^*.

Thus, T_0^ϵ denotes the set of all translated images of ϵT of the form $\epsilon(kL + \overline{T})$, $k \in \{0, 1, 2, \cdots\}$. Set

$$T^\epsilon = R^\epsilon \cap T_0^\epsilon.$$

Hence, the perforated thin domain is given by

$$R_p^\epsilon = R^\epsilon \setminus \overline{T^\epsilon}.$$

We assume that the holes do not intersect the boundary of R^ϵ. Thus, we assume that for every ϵ considered, the interval $(0,1)$ is a finite union of segments of length ϵL_1, that is, there exists an integer N_ϵ such that $\epsilon L_1(N_\epsilon + 1) = 1$ and

$$(0,1) = \mathrm{Int}\Big\{\bigcup_{k=0}^{N_\epsilon} [\epsilon k L_1, \epsilon L_1(k+1)]\Big\}$$

Notice that, with the assumptions above, we can write the perforated thin domain as

$$R_p^\epsilon = \mathrm{Int}\Big(\overline{R_-^\epsilon \cup R_{p+}^\epsilon}\Big),$$

where

$$R_-^\epsilon = \Big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1),\ -\epsilon\, h\Big(\frac{x}{\epsilon^\alpha}\Big) < y < -\epsilon h_0\Big)\Big\},$$

$$R_{p+}^\epsilon = \mathrm{Int}\Big(\bigcup_{k=0}^{N_\epsilon} \epsilon\big(kL_1 + \overline{Y_p^*}\big)\Big).$$

A model of this kind of thin domains is depicted in Figure 3.6.

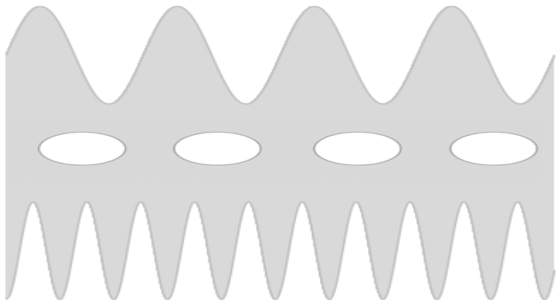


Figure 3.6: Perforated thin domain with doubly oscillatory boundary.

In order to analyze the problem (3.0.1) we first perform the change of variables $(x,y) \to (x, y/\epsilon)$, which transforms the domain R_p^ϵ into the domain Ω_p^ϵ

$$\Omega_p^\epsilon = \Big\{(x_1,x_2) \in \mathbb{R}^2 \mid (x_1, \epsilon x_2) \in R_p^\epsilon\Big\}. \tag{3.1.108}$$

Under this transformation, we obtain the equivalent linear elliptic problem

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f^\epsilon & \text{in } \Omega_p^\epsilon, \\ \dfrac{\partial u^\epsilon}{\partial x_1}\mu_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2}\mu_2^\epsilon = 0 & \text{on } \partial\Omega_p^\epsilon, \end{cases} \tag{3.1.109}$$

where $f \in L^2(0,1)$ and $\mu^\epsilon = (\mu_1^\epsilon, \mu_2^\epsilon)$ is the outward unit normal to $\partial\Omega_p^\epsilon$.

Observe that again, the domain Ω_p^ϵ consists on two parts clearly different: one of them is a highly oscillating domain Ω_-^ϵ and the other one is a perforated domain Ω_{p+}^ϵ with oscillations of order ϵ at the upper boundary. So, we have

$$\Omega_p^\epsilon = \mathrm{Int}\Big(\overline{\Omega_-^\epsilon \cup \Omega_{p+}^\epsilon}\Big),$$

where

$$\Omega_-^\epsilon = \Big\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ -h\Big(\frac{x_1}{\epsilon^\alpha}\Big) < y < -h_0\Big)\Big\},$$

$$\Omega_{p+}^\epsilon = \mathrm{Int}\Big(\bigcup_{k=0}^{N_\epsilon} \big(\epsilon kL_1 + \overline{Y_{\epsilon p}^*}\big)\Big),$$

with $Y^*_{\epsilon p} = \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1/\epsilon, x_2) \in Y^*_p\}$.

Now, in order to apply the techniques introduced in the previous sections we only need to construct an appropriate extension operator Q_ϵ which allows us to transform integrals over Ω^ϵ_p into integrals over the following non perforated domain

$$\widetilde{\Omega}^\epsilon = \Big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ -h(x_1/\epsilon^\alpha) < x_2 < g_1\Big\}.$$

In order to do so, as is usual in homogenization of reticulated structures (see e.g. [52]), we first define the extension in the unit cell Y^*_p and then, we derive the corresponding extension operator on Ω^ϵ_p using the periodicity of the holes and rescaling the extension of the cell to the whole perforated domain. Finally, we use the reflection procedure introduced in Lemma 3.1.2 to obtain the required extension operator.

Lemma 3.1.15. *With the notation above, there exists an extension operator*

$$Q_\epsilon \in \mathcal{L}(L^2(\Omega^\epsilon_p), L^2(\widetilde{\Omega}^\epsilon)) \cap \mathcal{L}(H^1(\Omega^\epsilon_p), H^1(\widetilde{\Omega}^\epsilon))$$

such that for any $\varphi \in H^1(\Omega^\epsilon_p)$,

$$\begin{aligned}
&||Q_\epsilon\varphi||_{L^2(\widetilde{\Omega}^\epsilon)} \le C||\varphi||_{L^2(\Omega^\epsilon_p)},\\
&\left\|\frac{\partial Q_\epsilon\varphi}{\partial x_1}\right\|_{L^2(\widetilde{\Omega}^\epsilon)} \le C\left\{\left\|\frac{\partial\varphi}{\partial x_1}\right\|_{L^2(\Omega^\epsilon_p)} + \frac{1}{\epsilon}\left\|\frac{\partial\varphi}{\partial x_2}\right\|_{L^2(\Omega^\epsilon_p)}\right\},\\
&\left\|\frac{\partial Q_\epsilon\varphi}{\partial x_2}\right\|_{L^2(\widetilde{\Omega}^\epsilon)} \le C\left\{\epsilon\left\|\frac{\partial\varphi}{\partial x_1}\right\|_{L^2(\Omega^\epsilon_p)} + \left\|\frac{\partial\varphi}{\partial x_2}\right\|_{L^2(\Omega^\epsilon_p)}\right\},
\end{aligned}$$

where C is a constant independent of ϵ.

Proof. Since the boundaries of the holes are smooth enough and the holes do not intersect the boundary of Y^* there exists an extension operator

$$S \in \mathcal{L}(L^2(Y^*_p), L^2(Y^*)) \cap \mathcal{L}(H^1(Y^*_p), H^1(Y^*))$$

such that for any $\varphi \in H^1(Y^*)$,

$$\begin{aligned}
&||S\varphi||_{L^2(Y^*)} \le C||\varphi||_{L^2(Y^*_p)},\\
&||\nabla S\varphi||_{[L^2(Y^*]^2} \le C||\nabla\varphi||_{[L^2(Y^*_p]^2}.
\end{aligned} \tag{3.1.110}$$

We refer the reader to [52] for the existence of this operator.

Now, we build an extension operator, S_ϵ, to extend the perforated domain Ω^ϵ_{p+} to the non perforated oscillating domain Ω^ϵ_+ given by

$$\Omega^\epsilon_+ = \Big\{(x_1, x_2) \in \mathbb{R}^2 \mid x \in (0,1),\ -h_0 < x_2 < g\Big(\frac{x_1}{\epsilon}\Big)\Big\} = \mathrm{Int}\Big(\bigcup_{k=0}^{N_\epsilon}\big(\epsilon k L_1 + \overline{Y^*_\epsilon}\big)\Big),$$

where $Y^*_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1/\epsilon, x_2) \in Y^*\}$.

Notice that, by the periodic structure of the domain Ω_+^ϵ we have

$$\int_{\Omega_+^\epsilon} \varphi \, dx_1 dx_2 = \int_{\bigcup_{k=0}^{N_\epsilon} \left(\epsilon k L_1 + Y_\epsilon^*\right)} \varphi \, dx_1 dx_2 = \sum_{k=0}^{N_\epsilon} \int_{\epsilon k L_1 + Y_\epsilon^*} \varphi \, dx_1 dx_2, \ \forall \varphi \in L^1(\Omega_+^\epsilon). \tag{3.1.111}$$

Then, to construct the extension operator S_ϵ we only need to know how to define it on every small cell $\epsilon k L_1 + Y_\epsilon^*$.

From the definition of $Y_{\epsilon p}^*$ it follows straightforward that for every (x_1, x_2) belonging to $kL_1 + Y_{\epsilon p}^*$ there exists $y_1 \in (0, L_1)$ such that

$$(x_1, x_2) = (\epsilon k L_1 + \epsilon y_1, x_2)$$

and $(y_1, x_2) \in Y_p^*$. Thus, we define the following family of functions

$$\varphi_{\epsilon k}(y_1, y_2) = \varphi(\epsilon k L_1 + \epsilon y_1, y_2), \quad \forall (y_1, y_2) \in Y_p^* \text{ and } k \in \{0, 1, \cdots, N_\epsilon\}.$$

Observe that $\varphi_{\epsilon k}$ is well defined in Y_p^* and $\varphi_{\epsilon k} \in H^1(Y_p^*)$.

Now, let us consider the following families of diffeomorphisms

$$\begin{aligned} D_k^\epsilon : \epsilon k L_1 + Y_\epsilon^* &\longrightarrow Y^* \\ (x_1, x_2) &\longrightarrow \Big(\tfrac{x_1 - \epsilon k L_1}{\epsilon}, x_2\Big). \end{aligned}$$

Then, with the notation above, using the extension operator on the cell, S, and taking into account that

$$\Omega_+^\epsilon = \mathrm{Int}\Big(\bigcup_{k=0}^{N_\epsilon} \big(\epsilon k L_1 + \overline{Y_\epsilon^*}\big)\Big),$$

we define $S_\epsilon \in \mathcal{L}(L^2(\Omega_{p+}^\epsilon),\ L^2(\Omega_+^\epsilon)) \cap \mathcal{L}(H^1(\Omega_{p+}^\epsilon),\ H^1(\Omega_+^\epsilon))$, acting on $\varphi \in L^2(\Omega_{p+}^\epsilon)$, as follows

$$(S_\epsilon \varphi)(x_1, x_2) = \big((S\varphi_{\epsilon k}) \circ D_k^\epsilon\big)(x_1, x_2) = (S\varphi_{\epsilon k})\big(\frac{x_1 - \epsilon L_1 k}{\epsilon}, x_2\big), \ \forall (x_1, x_2) \in \epsilon k L_1 + Y_\epsilon^*.$$

Moreover, let $\varphi \in H^1(\Omega_{p+}^\epsilon)$, then S_ϵ satisfies the following inequalities

$$\begin{aligned} &||S_\epsilon \varphi||_{L^2(\Omega_+^\epsilon)} \le C ||\varphi||_{L^2(\Omega_{p+}^\epsilon)}, \\ &\left\| \frac{\partial S_\epsilon \varphi}{\partial x_1} \right\|_{L^2(\Omega_+^\epsilon)} \le C \left\{ \left\| \frac{\partial \varphi}{\partial x_1} \right\|_{L^2(\Omega_{p+}^\epsilon)} + \frac{1}{\epsilon} \left\| \frac{\partial \varphi}{\partial x_2} \right\|_{L^2(\Omega_{p+}^\epsilon)} \right\}, \\ &\left\| \frac{\partial S_\epsilon \varphi}{\partial x_2} \right\|_{L^2(\Omega_+^\epsilon)} \le C \left\{ \epsilon \left\| \frac{\partial \varphi}{\partial x_1} \right\|_{L^2(\Omega_{p+}^\epsilon)} + \left\| \frac{\partial \varphi}{\partial x_2} \right\|_{L^2(\Omega_{p+}^\epsilon)} \right\}, \end{aligned} \tag{3.1.112}$$

which follows from (3.1.111) and taking into account that S_ϵ verifies the inequalities below due to (3.1.110)

$$\int_{\epsilon k L_1 + Y_\epsilon^*} |S_\epsilon \varphi|^2 dx_1 dx_2 = \epsilon \int_{Y^*} |S\varphi_{\epsilon k}|^2 dy_1 dy_2$$

$$
\begin{aligned}
&\leq C\epsilon \int_{Y_p^*} |\varphi_{\epsilon k}|^2 dy_1 dy_2 = C \int_{\epsilon k L_1 + Y_{\epsilon p}^*} |\varphi|^2 dx_1 dx_2, \\
\int_{\epsilon k L_1 + Y_\epsilon^*} |\frac{\partial S_\epsilon \varphi}{\partial x_1}|^2 dx_1 dx_2 &= \epsilon \int_{Y^*} |\frac{1}{\epsilon}\frac{\partial S\varphi_{\epsilon k}}{\partial y_1}|^2 dy_1 dy_2 \\
&\leq C\frac{1}{\epsilon}\int_{Y_p^*} \left\{ |\frac{\partial \varphi_{\epsilon k}}{\partial y_1}|^2 + |\frac{\partial \varphi_{\epsilon k}}{\partial y_2}|^2 \right\} dy_1 dy_2 \\
&= C \int_{\epsilon k L_1 + Y_{\epsilon p}^*} \left\{ |\frac{\partial \varphi}{\partial x_1}|^2 + \frac{1}{\epsilon^2}|\frac{\partial \varphi}{\partial y_2}|^2 \right\} dx_1 dx_2, \\
\int_{\epsilon k L_1 + Y_\epsilon^*} |\frac{\partial S_\epsilon \varphi}{\partial x_2}|^2 dx_1 dx_2 &= \epsilon \int_{Y^*} |\frac{\partial S\varphi_{\epsilon k}}{\partial y_2}|^2 dy_1 dy_2 \\
&\leq C\epsilon \int_{Y_p^*} \left\{ |\frac{\partial \varphi_{\epsilon k}}{\partial y_1}|^2 + |\frac{\partial \varphi_{\epsilon k}}{\partial y_2}|^2 \right\} dy_1 dy_2 \\
&= C \int_{\epsilon k L_1 + Y_{\epsilon p}^*} \left\{ \epsilon^2 |\frac{\partial \varphi}{\partial x_1}|^2 + |\frac{\partial \varphi}{\partial y_2}|^2 \right\} dx_1 dx_2.
\end{aligned}
$$

Thus, denoting by Ω^ϵ the doubly oscillating domain without holes given by

$$\Omega^\epsilon = \mathrm{Int}\Big(\overline{\Omega_-^\epsilon \cup \Omega_+^\epsilon}\Big) = \Big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ -h\Big(\frac{x_1}{\epsilon^\alpha}\Big) < x_2 < \epsilon\, g\Big(\frac{x_1}{\epsilon}\Big)\Big\},$$

we define the extension operator $Q_\epsilon^1 \in \mathcal{L}(L^2(\Omega_p^\epsilon), L^2(\Omega^\epsilon)) \cap \mathcal{L}(H^1(\Omega_p^\epsilon), H^1(\Omega^\epsilon))$, acting on $\varphi \in H^1(\Omega_p^\epsilon)$, as follows

$$(Q_\epsilon^1 \varphi)(x_1, x_2) = \begin{cases} \varphi(x_1, x_2) & (x_1, x_2) \in \Omega_-^\epsilon \\ (S^\epsilon \varphi)(x_1, x_2) & (x_1, x_2) \in \Omega_+^\epsilon, \end{cases}$$

Furthermore, it is obvious from (3.1.112) that Q_ϵ^1 satisfies that

$$
\begin{aligned}
&||Q_\epsilon^1 \varphi||_{L^p(\Omega^\epsilon)} \leq C ||\varphi||_{L^p(\Omega_p^\epsilon)}, \\
&\left\| \frac{\partial Q_\epsilon^1 \varphi}{\partial x_1} \right\|_{L^p(\Omega^\epsilon)} \leq C \left\{ \left\| \frac{\partial \varphi}{\partial x_1} \right\|_{L^p(\Omega_p^\epsilon)} + \frac{1}{\epsilon} \left\| \frac{\partial \varphi}{\partial x_2} \right\|_{L^p(\Omega_p^\epsilon)} \right\}, \\
&\left\| \frac{\partial Q_\epsilon^1 \varphi}{\partial x_2} \right\|_{L^p(\Omega^\epsilon)} \leq C\epsilon \left\| \frac{\partial \varphi}{\partial x_1} \right\|_{L^p(\Omega_p^\epsilon)} + C \left\| \frac{\partial \varphi}{\partial x_2} \right\|_{L^p(\Omega_p^\epsilon)}.
\end{aligned}
\tag{3.1.113}
$$

Finally, since $Q_\epsilon^1 \varphi \in H^1(\Omega^\epsilon)$ for any $\varphi \in H^1(\Omega_p^\epsilon)$ we define the required extension operator as $Q_\epsilon = P_\epsilon \circ Q_\epsilon^1$ where P_ϵ is the operator introduced in Lemma 3.1.2. Then, we have

$$Q_\epsilon \in \mathcal{L}(L^2(\Omega_p^\epsilon), L^2(\widetilde{\Omega}^\epsilon)) \cap \mathcal{L}(H^1(\Omega_p^\epsilon), H^1(\widetilde{\Omega}^\epsilon))$$

and taking into account that P_ϵ and Q_ϵ^1 verify (3.1.10) and (3.1.113) respectively we obtain the following inequalities for any $\varphi \in H^1(\Omega_p^\epsilon)$

$$
\begin{aligned}
&||Q_\epsilon \varphi||_{L^p(\widetilde{\Omega}^\epsilon)} = ||P_\epsilon(Q_\epsilon^1 \varphi)||_{L^p(\widetilde{\Omega}^\epsilon)} \leq C ||Q_\epsilon^1 \varphi||_{L^p(\Omega^\epsilon)} \leq C ||\varphi||_{L^p(\Omega_p^\epsilon)}, \\
&\left\| \frac{\partial Q_\epsilon \varphi}{\partial x_1} \right\|_{L^p(\widetilde{\Omega}^\epsilon)} = \left\| \frac{\partial P_\epsilon(Q_\epsilon^1 \varphi)}{\partial x_1} \right\|_{L^p(\widetilde{\Omega}^\epsilon)} \leq C \left\{ \left\| \frac{\partial Q_\epsilon^1 \varphi}{\partial x_1} \right\|_{L^p(\Omega^\epsilon)} + \frac{1}{\epsilon} \left\| \frac{\partial Q_\epsilon^1 \varphi}{\partial x_2} \right\|_{L^p(\Omega^\epsilon)} \right\},
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\left\{\left\|\frac{\partial\varphi}{\partial x_1}\right\|_{L^2(\Omega_p^\epsilon)}+\frac{1}{\epsilon}\left\|\frac{\partial\varphi}{\partial x_2}\right\|_{L^2(\Omega_p^\epsilon)}\right\},\\
\left\|\frac{\partial Q_\epsilon\varphi}{\partial x_2}\right\|_{L^p(\widetilde{\Omega}^\epsilon)}=\left\|\frac{\partial P_\epsilon(Q_\epsilon^1\varphi)}{\partial x_2}\right\|_{L^p(\widetilde{\Omega}^\epsilon)}&\leq C\left\|\frac{\partial Q_\epsilon^1\varphi}{\partial x_2}\right\|_{L^p(\Omega^\epsilon)}\\
&\leq C\left\{\epsilon\left\|\frac{\partial\varphi}{\partial x_1}\right\|_{L^2(\Omega_p^\epsilon)}+\left\|\frac{\partial\varphi}{\partial x_2}\right\|_{L^2(\Omega_p^\epsilon)}\right\},
\end{aligned}
$$

which ends the proof. □

Once we have proved the existence of a suitable extension operator for the perforated domains Ω_p^ϵ it is obvious that the same method as the previous subsections applies. Then, we state the corresponding homogenization result without giving the proof.

Theorem 3.1.16. *Let u^ϵ be the unique solution of (3.1.109). Then, there exists $u_0 \in H^1(0,1)$ such that, if Q_ϵ is the extension operator constructed in Lemma 3.1.15 one has*

$$
\|Q_\epsilon u^\epsilon - u_0\|_{L^2(\widetilde{\Omega}^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0. \tag{3.1.114}
$$

$u_0 \in H^1(0,1)$ is the unique weak solution of the following Neumann problem

$$
\begin{cases}
-\dfrac{\hat{q}}{\frac{|Y_p^*|}{L_1}+p}u_{0xx}(x)+u_0(x)=f(x), \quad x\in(0,1),\\
u_0'(0)=u_0'(1)=0,
\end{cases}
$$

where the homogenized constant coefficients are defined by

$$
\begin{aligned}
\hat{q} &= \frac{1}{L_1}\int_{Y_p^*}\left\{1-\frac{\partial X}{\partial y_1}(y_1,y_2)\right\}dy_1dy_2,\\
p &= \int_0^1 h(s)\,ds - h_0,
\end{aligned}
$$

Y_p^* is the representative cell defined in (3.1.107) and X is the unique solution (up to constants) which is L_1-periodic in the first variable, of the problem

$$
\begin{cases}
-\Delta X = 0 \text{ in } Y_p^*,\\
\dfrac{\partial X}{\partial N}=0 \text{ on } B_2,\\
\dfrac{\partial X}{\partial N}=-\dfrac{g'(y_1)}{\sqrt{1+g'(y_1)^2}} \text{ on } B_1,\\
\dfrac{\partial X}{\partial N}=N_1 \text{ on } \partial T,
\end{cases}
$$

where B_1 is the upper boundary and B_2 is the lower boundary of ∂Y^*.

Furthermore, let w^ϵ be the solution of the problem (3.0.1). Then the following convergence holds

$$\lim_{\epsilon\to 0} |||w^\epsilon - u_0 - \kappa^\epsilon|||_{H^1(R^\epsilon)} = 0,$$

where κ^ϵ is the first-order corrector given by

$$\kappa^\epsilon(x,y) = \begin{cases} -\epsilon X\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}\right)\frac{\partial u_0}{\partial x}, & \text{if } (x,y) \in R^\epsilon_{p+}, \\ -u_0 + v^\epsilon(x,y), & \text{if } (x,y) \in R^\epsilon_{-1}. \end{cases}$$

with $v^\epsilon(x,y) = v^\epsilon_n(x, y/\epsilon)$ where v^ϵ_n is the solution of (3.1.85) *with Q^ϵ_n given by* (3.1.34).

Remark 3.1.17. *We may also consider that the oscillatory part at the bottom boundary has also some holes. That is, if we consider the lower cell:*

$$Y^*_- = \{(y_1, y_2) : \ 0 < y_1 < L_2, \ -h(y_1) < y_2 < -h_0\}$$

(see Remark 3.1.8) and we perforate a hole on it, that is, we consider a set T_- so that $\bar{T}_- \subset Y^*_-$ and consider

$$Y^*_{-,p} = Y^*_- \setminus \bar{T}_-$$

and we construct the thin domain allowing both holes in the upper part and at the lower boundary, then the limit problem will be:

$$\begin{cases} -\dfrac{\hat{q}}{\frac{|Y^*_p|}{L_1} + \frac{|Y^*_{-,p}|}{L_2}} u_{0xx}(x) + u_0(x) = f(x), & x \in (0,1), \\ u_0'(0) = u_0'(1) = 0, \end{cases}$$

where q is given exactly as in Theorem 3.1.16.

3.2. Fast and weak boundary oscillations in thin domains

In the present section we analyze the behavior of solutions of problem (3.0.1) posed in a 2-dimensional thin domain with order of thickness ϵ which presents a highly oscillatory behavior at the bottom boundary and a weak oscillatory behavior at the top boundary. Namely, we combine techniques introduced in the previous section and the unfolding operator method presented in Chapter 1 to obtain the homogenized limit problem associated to (3.0.1) where the thin domain R^ϵ is given by

$$R^\epsilon = \Big\{(x,y) \in \mathbb{R}^2 \mid x \in (0,1), \ -\epsilon\, h\Big(\frac{x}{\epsilon^\alpha}\Big) < y < \epsilon\, g\Big(\frac{x}{\epsilon^\beta}\Big)\Big\}, \tag{3.2.1}$$

where $\alpha > 1, 0 < \beta < 1$ and $g, h : \mathbb{R} \to \mathbb{R}$ satisfies hypothesis **(H)** from the beginning of the chapter.

Notice that, in this case, the difference between the order of the oscillations at the top and bottom boundary is much larger than the previous cases. In fact, the lower boundary continues to present an extremely high oscillatory behavior, the period has order ϵ^α with $\alpha > 1$, while the roughness of the the upper boundary is very slight, the period of these oscillations has order ϵ^β with $0 < \beta < 1$.

Indeed, as in the previous section, the strong roughness at the bottom boundary allows us to split the domain in such a way that we can separate the different scales and to construct suitable oscillating test functions.

Therefore, the domain R^ϵ is decomposed in two parts: one of them, R^ϵ_-, presents fast oscillations and the other one, R^ϵ_+, is a weakly oscillating thin domain, that is,

$$\begin{aligned} R^\epsilon_- &= \{(x,y) \in \mathbb{R}^2 | \, x \in (0,1), \, -\epsilon h(x/\epsilon^\alpha) < y < -\epsilon h_0\} \\ R^\epsilon_+ &= \{(x,y) \in \mathbb{R}^2 | \, x \in (0,1), \, -\epsilon h_0 < y < \epsilon g(x_1/\epsilon^\beta)\}. \end{aligned}$$

Observe that,

$$R^\epsilon = \mathrm{Int}\Big(\overline{R^\epsilon_+ \bigcup R^\epsilon_-}\Big).$$

In this situation, we denote the restriction of solutions to each part by

$$w^\epsilon_+ := w^\epsilon|_{R^\epsilon_+} \text{ and } w^\epsilon_- := w^\epsilon|_{R^\epsilon_-}.$$

Note that, to obtain the homogenized limit problem we first apply the corresponding unfolding operator to each open set and then, using the techniques introduced in this chapter we construct appropriate oscillating test functions which allow us to pass to the limit.

Thus, we denote the unfolding operator associated to the lower part by $\mathcal{T}_{\epsilon_-}$ and to the upper part by $\mathcal{T}_{\epsilon_+}$. Notice that, since $\mathcal{T}_{\epsilon_-}$ is associated to the strong oscillations it is defined and has the same properties as the unfolding of Section 1.4. Moreover, $\mathcal{T}_{\epsilon_+}$ is associated to the weak oscillations and it satisfies the results introduced in Section 1.3.

Recall that the variational formulation of (3.0.1) is: find $w^\epsilon \in H^1(R^\epsilon)$ such that

$$\int_{R^\epsilon} \Big\{ \frac{\partial w^\epsilon}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial w^\epsilon}{\partial y}\frac{\partial \varphi}{\partial y} + w^\epsilon \varphi \Big\} dxdy = \int_{R^\epsilon} f^\epsilon \varphi \, dxdy, \quad \forall \varphi \in H^1(R^\epsilon), \tag{3.2.2}$$

We can now state the homogenization result.

Theorem 3.2.1. *Let w^ϵ be the solution of problem (3.0.1) with R^ϵ given by* (3.2.1). *Assume that the non-homogeneous term is uniformly bounded, $|||f^\epsilon|||_{L^2(R^\epsilon)} \leq C$ and that there exists $\hat{f} \in L^2(0,1)$ such that*

$$\hat{f}^\epsilon \overset{\epsilon \to 0}{\rightharpoonup} \hat{f}, \; w - L^2(0,1), \tag{3.2.3}$$

where

$$\hat{f}^\epsilon(x) \equiv \frac{1}{\epsilon} \int_{-\epsilon h(x/\epsilon^\alpha)}^{\epsilon g(x/\epsilon^\beta)} f^\epsilon(x,y) \, dy.$$

Then, there exists $u_0 \in H^1(0,1)$ such that $|||w^\epsilon - u_0|||_{L^2(R^\epsilon)} \to 0$ and it is the unique solution of the following Neumann problem:

$$\begin{cases} -\dfrac{1}{\mathcal{M}\big(\frac{1}{g+h_0}\big)\big(\mathcal{M}(g)+\mathcal{M}(h)\big)} u_{0xx} + u_0 = \dfrac{\hat{f}}{\mathcal{M}(g)+\mathcal{M}(h)}, & x \in (0,1) \\ u_0'(0) = u_0'(1) = 0. \end{cases} \tag{3.2.4}$$

where the operator $\mathcal{M}$ applied to a periodic function is the average of the function over one period, see Notation Section.

Proof. First of all, note that taking w^ϵ as a test function in the variational formulation (3.2.2) and using that $|||f^\epsilon|||_{L^2(R^\epsilon)} \le C$, we easily obtain the following a priori estimate

$$|||w^\epsilon|||_{H^1(R^\epsilon)} \le C, \tag{3.2.5}$$

where the constant C does not depend on ϵ.

From the a priori estimate (3.2.5) and taking into account Proposition 1.1.14 and Theorem 1.3.1 we may ensure that there exist $u_+ \in H^1(0,1)$ and $u_1 \in L^2((0,1); H^1_\#(Y^*_+))$ with $\frac{\partial u_1}{\partial y_2} = 0$ such that, up to subsequences:

$$\begin{aligned} &\mathcal{T}_{\epsilon_+}(w^\epsilon_+) \overset{\epsilon\to 0}{\longrightarrow} u_+ \quad \mathrm{w} - L^2\big((0,1); H^1(Y^*_+)\big),\\ &\mathcal{T}_{\epsilon_+}(w^\epsilon_+) \overset{\epsilon\to 0}{\longrightarrow} u_+ \quad \mathrm{s} - L^2\big((0,1)\times Y^*_+)\big),\\ &\mathcal{T}_{\epsilon_+}\Big(\frac{\partial w^\epsilon_+}{\partial x_1}\Big) \overset{\epsilon\to 0}{\rightharpoonup} \frac{\partial u_+}{\partial x_1} + \frac{\partial u_1}{\partial y_1} \quad \mathrm{w} - L^2\big((0,1)\times Y^*_+\big), \end{aligned} \tag{3.2.6}$$

where Y^*_+ is the reference cell given by

$$Y^*_+ = \{(y_1, y_2) \in \mathbb{R}^2 \mid 0 < y_1 < L_1,\ -h_0 < y_2 < g(y_1)\}.$$

Also, using the same reasoning as in the proof of Theorem 1.4.3, see (1.4.6)-(1.4.15), we obtain that there is a function $u_- \in L^2(0,1)$ such that

$$\begin{aligned} &\mathcal{T}_{\epsilon_-}(w^\epsilon_-) \overset{\epsilon\to 0}{\rightharpoonup} u_- \quad \mathrm{w} - L^2\big((0,1)\times Y^*_-\big),\\ &\mathcal{T}_{\epsilon_-}\Big(\frac{\partial w^\epsilon_-}{\partial x_1}\Big) \overset{\epsilon\to 0}{\rightharpoonup} 0 \quad \mathrm{w} - L^2\big((0,1)\times Y^*_-\big), \end{aligned} \tag{3.2.7}$$

where Y^*_- is the basic cell associated to the lower part of the thin domain

$$Y^*_- = \{(y_1, y_2) \in \mathbb{R}^2 \mid 0 < y_1 < L_1,\ -h(y_1) < y_2 < -h_0\}.$$

Now we are going to check that

$$u_+(x) = u_-(x) \ \text{ for a.e. } x \in (0,1),$$

and taking $u_0 = u_+$ we will prove

$$\lim_{\epsilon\to 0} |||w^\epsilon - u_0|||_{L^2(R^\epsilon)} = 0. \tag{3.2.8}$$

To do so, we use the rescaling operator introduced in Subsection 1.4.1. Then, we have

$$\Pi_\epsilon(w^\epsilon)(x, x_2) = w^\epsilon(x_1, \epsilon x_2), \quad \forall (x, x_2) \in R_0,$$

where $R_0 = \{(x, x_2) \in \mathbb{R}^2 | \, x \in (0,1),\ -h_0 < x_2 < g_0\}$.

Then, from the a priori estimate (3.2.5) we have that there exists $u_0 \in H^1(R_0)$ such that, up to subsequences,

$$\begin{aligned} &\Pi_\epsilon(w^\epsilon) \overset{\epsilon\to 0}{\rightharpoonup} u_0 \quad \mathrm{w}-H^1(R_0), \\ &\Pi_\epsilon\Big(\frac{\partial w^\epsilon}{\partial x}\Big) \overset{\epsilon\to 0}{\rightharpoonup} \frac{\partial u_0}{\partial x} \quad \mathrm{w}-L^2(R_0), \\ &\Pi_\epsilon\Big(\frac{\partial w^\epsilon}{\partial y}\Big) \overset{\epsilon\to 0}{\longrightarrow} 0 \quad \mathrm{s}-L^2(R_0). \end{aligned} \tag{3.2.9}$$

This implies that u_0 does not depend on the second variable and therefore $u_0 \in H^1(0,1)$.

To get the strong convergence (3.2.8) we argue as follows. On one hand, from convergence (3.2.9) one has

$$\begin{aligned} |||\Pi_\epsilon(w^\epsilon)|_{x_2=0} - u_0|||^2_{L^2(R^\epsilon)} &= \frac{1}{\epsilon}\int_0^1 \int_{-\epsilon h(x/\epsilon^\alpha)}^{\epsilon g(x/\epsilon^\beta)} |\Pi_\epsilon(w^\epsilon)|_{x_2=0} - u_0|^2 \, dydx \\ &\le C\,||\Pi_\epsilon(w^\epsilon)|_{x_2=0} - u_0||^2_{L^2(0,1)} \overset{\epsilon\to 0}{\longrightarrow} 0. \end{aligned} \tag{3.2.10}$$

On the other hand

$$\begin{aligned} |||w^\epsilon - \Pi_\epsilon(w^\epsilon)|_{x_2=0}|||^2_{L^2(R^\epsilon)} &= \frac{1}{\epsilon}\int_0^1 \int_{-\epsilon h(x/\epsilon^\alpha)}^{\epsilon g(x/\epsilon^\beta)} |w^\epsilon(x,y) - w^\epsilon(x,0)|^2 \, dydx \\ &\le \frac{1}{\epsilon}\int_0^1 \int_{-\epsilon h(x/\epsilon^\alpha)}^{\epsilon g(x/\epsilon^\beta)} \Big(\int_0^y \Big|\frac{\partial w^\epsilon}{\partial y}(x,s)\Big|^2 ds\Big)|y| dydx \\ &\le \epsilon C\Big|\Big|\Big|\frac{\partial w^\epsilon}{\partial y}\Big|\Big|\Big|^2_{L^2(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0. \end{aligned} \tag{3.2.11}$$

Hence, using the convergences above we get (3.2.8)

$$|||w^\epsilon - u_0|||_{L^2(R^\epsilon)} \le |||w^\epsilon - \Pi_\epsilon(w^\epsilon)|_{x_2=0}|||_{L^2(R^\epsilon)} + |||\Pi_\epsilon(w^\epsilon)|_{x_2=0} - u_0|||_{L^2(R^\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0.$$

Now, by property vi) of Proposition 1.1.4 we have

$$\|\mathcal{T}_{\epsilon_-}(w^\epsilon_-) - \mathcal{T}_{\epsilon_-}(u_0)\|_{L^2\left((0,1)\times Y^*_-\right)} \le C|||w^\epsilon - u_0|||_{L^2(R^\epsilon_-)},$$

$$\|\mathcal{T}_{\epsilon_+}(w^\epsilon_+) - \mathcal{T}_{\epsilon_+}(u_0)\|_{L^2\left((0,1)\times Y^*_+\right)} \le C|||w^\epsilon - u_0|||_{L^2(R^\epsilon_+)}.$$

Therefore, from convergence (3.2.8) and taking into account the following convergences, see Proposition 1.1.10 in Chapter 1,

$$\mathcal{T}_{\epsilon_-}(u_0) \overset{\epsilon\to 0}{\longrightarrow} u_0 \quad \mathrm{s}-L^2\big((0,1)\times Y^*_-\big)$$

$$\mathcal{T}_{\epsilon_+}(u_0) \overset{\epsilon\to 0}{\longrightarrow} u_0 \quad \mathrm{s}-L^2\big((0,1)\times Y^*_+\big),$$

we obtain

$$\mathcal{T}_{\epsilon_-}(w^\epsilon_-) \overset{\epsilon\to 0}{\longrightarrow} u_0 \quad \mathrm{s}-L^2\big((0,1)\times Y^*_-\big),$$

$$\mathcal{T}_{\epsilon+}(w_+^\epsilon) \stackrel{\epsilon\to 0}{\longrightarrow} u_0 \quad \text{s} - L^2\big((0,1)\times Y_+^*\big).$$

Then, from now on, we denote by u_0 the limit of $\mathcal{T}_{\epsilon-}(w_-^\epsilon)$ and the limit of $\mathcal{T}_{\epsilon+}(w_+^\epsilon)$.

Now, since the variational formulation (3.2.2) can be written as

$$\frac{1}{\epsilon}\Big\{\int_{R_+^\epsilon}\frac{\partial w^\epsilon}{\partial x}\frac{\partial \phi}{\partial x}\,dxdy + \int_{R_-^\epsilon}\frac{\partial w^\epsilon}{\partial x}\frac{\partial \phi}{\partial x}\,dxdy + \int_{R_+^\epsilon} w^\epsilon\phi\,dxdy + \int_{R_-^\epsilon} w^\epsilon\phi\,dxdy\Big\}$$
$$= \int_0^1 \hat{f}^\epsilon\phi dxdy, \quad \forall\phi\in H^1(0,1),$$

we apply the unfolding operators $\mathcal{T}_{\epsilon-}$ and $\mathcal{T}_{\epsilon+}$ to the corresponding terms and taking into account convergences (3.2.3), (3.2.6) and (3.2.7) we obtain at the limit

$$\frac{1}{L_1}\int_{(0,1)\times Y_+^*}\Big(\frac{\partial u_0}{\partial x}+\frac{\partial u_1}{\partial y_1}\Big)\frac{\partial\phi}{\partial x}\,dxdy_1dy_2 + \int_0^1\Big(\mathcal{M}(g)+\mathcal{M}(h)\Big)u_0\phi\,dx = \int_0^1\hat{f}\phi\,dx. \tag{3.2.12}$$

Finally, to characterize $\frac{\partial u_1}{\partial y_1}$ we construct suitable oscillating test functions in the same way as we have made in the previous section.

Then, we consider the partition of $(0,1)$ given by the points $\{\gamma_{0,\epsilon},\gamma_{1,\epsilon},\dots,\gamma_{N_\epsilon+1,\epsilon}\}$ associated to the function h as described in Section 3.1.2, item **(d)**.

We define now the test functions as follows. Let $\phi\in\mathcal{D}(0,1)$ and $\psi\in H^1_\#(0,L_1)$, we define $\varphi^\epsilon\in H^1(R^\epsilon)$ as follows

$$\varphi^\epsilon(x,y)=\begin{cases} v^\epsilon(x), & (x,y)\in R_+^\epsilon,\\ W^\epsilon(x,y), & (x,y)\in R_-^\epsilon\end{cases} \tag{3.2.13}$$

where $v^\epsilon(x)=\epsilon^\beta\phi(x)\psi(x/\epsilon^\beta)$, and $W^\epsilon(x,y)=W_n^\epsilon(x,y/\epsilon)$, for $(x,y/\epsilon)\in Q_n^\epsilon$. Recall that Q_n^ϵ are the rectangles given by

$$Q_n^\epsilon=\{(x_1,x_2)\,|\,\gamma_{n,\epsilon}<x_1<\gamma_{n+1,\epsilon},\,-h_1<x_2<-h_0\},\quad n=1,2,\dots,N_\epsilon.$$

and in this case W_n^ϵ is the solution of the following auxiliary problem

$$\begin{cases} \dfrac{\partial^2 W_n^\epsilon}{\partial x_1^2}+\dfrac{1}{\epsilon^2}\dfrac{\partial^2 W_n^\epsilon}{\partial x_2^2}=0, & \text{in } Q_n^\epsilon\\ \dfrac{\partial W_n^\epsilon}{\partial\nu}=0, & \text{on } \partial Q_n^\epsilon\backslash\Gamma_n^\epsilon\\ W_n^\epsilon(x_1,x_2)=v^\epsilon(x_1), & \text{on } \Gamma_n^\epsilon\end{cases}$$

where Γ_n^ϵ is the top of the rectangle, that is, $\Gamma_n^\epsilon=\{(x_1,h_0):\gamma_{n,\epsilon}\le x_1\le\gamma_{n+1,\epsilon}\}$.

From Lemma 3.1.4 we have

$$\Big|\Big|\frac{\partial W_n^\epsilon}{\partial x_1}\Big|\Big|^2_{L^2(Q_n^\epsilon)}+\frac{1}{\epsilon^2}\Big|\Big|\frac{\partial W_n^\epsilon}{\partial x_2}\Big|\Big|^2_{L^2(Q_n^\epsilon)}\le C\epsilon^{\alpha-1}\Big|\Big|\frac{\partial v^\epsilon}{\partial x_1}\Big|\Big|^2_{L^2(\gamma_{n,\epsilon},\gamma_{n+1,\epsilon})},$$

where the constant C does not depend on ϵ.

Consequently, since $R^\epsilon_- = \cup_{n=0}^{N_\epsilon} Q^\epsilon_n \cap R^\epsilon$ and using the change of variables $(x, y) \to (x, y/\epsilon)$ it follows that

$$\Big|\Big|\Big| \frac{\partial W^\epsilon}{\partial x} \Big|\Big|\Big|^2_{L^2(R^\epsilon_-)} + \Big|\Big|\Big| \frac{\partial W^\epsilon}{\partial y} \Big|\Big|\Big|^2_{L^2(R^\epsilon_-)} \le C\epsilon^{\alpha - 1} \Big\| \frac{\partial v^\epsilon}{\partial x} \Big\|^2_{L^2(0,1)}, \tag{3.2.14}$$

Furthermore, in view of definition (3.2.13) φ^ϵ satisfies

$$\big|\varphi^\epsilon(x, y) - v^\epsilon(x)\big| = \big|\varphi^\epsilon(x, y) - \varphi^\epsilon(x, -\epsilon h_0)\big| = \Big| \int_y^{-\epsilon h_0} \frac{\partial \varphi^\epsilon}{\partial y}(x, s)\, ds \Big|, \ \forall (x, y) \in R^\epsilon_-.$$

Then, taking into account bounds (3.2.14), v^ϵ does not depend on y and using the same arguments as (3.1.38) we obtain that

$$|||\varphi^\epsilon - v^\epsilon|||_{L^2(R^\epsilon)} \overset{\epsilon \to 0}{\longrightarrow} 0.$$

Consequently, since $|||v^\epsilon|||_{L^2(R^\epsilon)} \to 0$ we get

$$|||\varphi^\epsilon|||_{L^2(R^\epsilon)} \overset{\epsilon \to 0}{\longrightarrow} 0. \tag{3.2.15}$$

Therefore, using the convergences above we will pass to the limit at the weak formulation (3.2.2) taking φ^ϵ as test function.

On one hand, taking into account (3.2.5), (3.2.13), (3.2.15) and (3.2.14) it is obvious that

$$\epsilon^{-1} \int_{R^\epsilon_-} \Big\{ \frac{\partial w^\epsilon}{\partial x} \frac{\partial \varphi^\epsilon}{\partial x} + \frac{\partial w^\epsilon}{\partial y} \frac{\partial \varphi^\epsilon}{\partial y} \Big\}\, dxdy + \epsilon^{-1} \int_{R^\epsilon} \Big\{ w^\epsilon \varphi^\epsilon - f^\epsilon \varphi^\epsilon \Big\}\, dxdy \overset{\epsilon \to 0}{\longrightarrow} 0. \tag{3.2.16}$$

On the other hand, since $v^\epsilon(x) = \epsilon^\beta \phi(x) \psi(x/\epsilon^\beta)$ it is easy to get the partial derivatives

$$\frac{\partial v^\epsilon}{\partial x} = \epsilon^\beta \frac{\partial \phi}{\partial x}(x) \psi\Big(\frac{x}{\epsilon^\beta}\Big) + \phi(x) \frac{\partial \psi}{\partial y_1}\Big(\frac{x}{\epsilon^\beta}\Big), \qquad \frac{\partial v^\epsilon}{\partial y} = 0.$$

Thus, using the basic properties of the unfolding operator we easily get

$$\begin{aligned}
&\mathcal{T}_{\epsilon+}(v^\epsilon) \to 0 \quad \text{s-}L^2((0,1) \times Y^*_+), \\
&\mathcal{T}_{\epsilon+}\Big(\frac{\partial v^\epsilon}{\partial x}\Big) \to \phi \frac{\partial \psi}{\partial y_1} \quad \text{s-}L^2((0,1) \times Y^*_+), \\
&\mathcal{T}_{\epsilon+}\Big(\frac{\partial v^\epsilon}{\partial y}\Big) = 0.
\end{aligned} \tag{3.2.17}$$

Hence, applying the unfolding operator $\mathcal{T}_{\epsilon_+}$ and due to convergences (3.2.17) and (3.2.6) we get

$$\epsilon^{-1} \int_{R^\epsilon_+} \Big\{ \frac{\partial w^\epsilon}{\partial x} \frac{\partial v^\epsilon}{\partial x} + \frac{\partial w^\epsilon}{\partial y} \frac{\partial v^\epsilon}{\partial x_2} \Big\}\, dxdy \to \frac{1}{L_1} \int_{(0,1) \times Y^*_+} \Big(\frac{\partial u_0}{\partial x} + \frac{\partial u_1}{\partial y_1} \Big) \phi \frac{\partial \psi}{\partial y_1}\, dx_1 dy_1 dy_2. \tag{3.2.18}$$

Hence, in view of (3.2.18) and (3.2.16), passing to the limit at the weak formulation (3.2.2) with φ^ϵ as test function we get

$$\int_{(0,1)\times Y_+^*} \Big(\frac{\partial u_0}{\partial x}(x) + \frac{\partial u_1}{\partial y_1}(x,y_1)\Big)\phi(x)\frac{\partial \psi}{\partial y_1}(z_1)\, dxdy_1dy_2 = 0,$$

for any $\phi \in \mathcal{D}(0,1)$ and $\psi \in H^1_\#(0,L_1)$. By density, this equality holds true for all $\psi \in L^2\big((0,1); H^1_\#(Y_+^*)\big)$ with $\frac{\partial \psi}{\partial y_2} = 0$

$$\int_{(0,1)\times Y_+^*} \Big(\frac{\partial u_0}{\partial x}(x) + \frac{\partial u_1}{\partial y_1}(x,y_1)\Big)\frac{\partial \psi}{\partial y_1}(y_1)\, dxdy_1dy_2 = 0.$$

Then, the same computations as in (1.3.10) lead to

$$\frac{\partial u_1}{\partial y_1} = \Big(-1 + \frac{1}{(h_0+g(y_1))\mathcal{M}\big(\frac{1}{h_0+g}\big)}\Big)\frac{\partial u_0}{\partial x}.$$

Replacing $\frac{\partial u_1}{\partial y_1}$ by this value in the equation (3.2.12) and with elementary computations, we obtain:

$$\int_{(0,1)} \frac{1}{\mathcal{M}\big(\frac{1}{g+h_0}\big)}\frac{\partial u_0}{\partial x_1}\frac{\partial \phi}{\partial x_1}\, dx_1dz_1dz_2 + \int_0^1 \big(\mathcal{M}(g)+\mathcal{M}(h)\big)u_0\phi\, dx_1 = \int_0^1 \hat{f}\phi\, dx_1. \tag{3.2.19}$$

From Lax-Milgram Theorem we know that u_0 is the unique solution of (3.2.19), which is the variational formulation of (3.2.4).

This completes the proof of Theorem 3.2.1. □

Remark 3.2.2. *In case the nonhomogeneous term $f^\epsilon(x,y) = f_0(x)$, then $\hat{f} = (\mathcal{M}(g)+\mathcal{M}(h))f_0$ and therefore the limit problem can be written as:*

$$\begin{cases} -\dfrac{1}{\mathcal{M}\big(\frac{1}{g+h_0}\big)\big(\mathcal{M}(g)+\mathcal{M}(h)\big)}u_{0xx} + u_0 = f_0, & x \in (0,1) \\ u_0'(0) = u_0'(1) = 0. \end{cases}$$

3.3. Thin domains with fast oscillations at the top and the bottom boundary.

In the present section we conclude the study initiated in the two previous sections on the homogenization of thin domains with doubly oscillatory boundaries where at least one of two boundaries, bottom or top boundary, presents a fast oscillatory behavior. Notice that in Section 3.1 we studied the case fast-resonant and in Section 3.2 we studied the case fast-weak. In the present section we address the case of fast-fast.

Hence, R^ϵ is given by

$$R^\epsilon = \Big\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ -\epsilon\, h(x_1/\epsilon^\alpha) < x_2 < \epsilon\, g(x_1/\epsilon^\beta)\Big\}, \quad \beta > 1, \alpha > 1. \tag{3.3.1}$$

Recall that $g, h : \mathbb{R} \to \mathbb{R}$ satisfy hypothesis **(H)**.

Observe that in this case both upper and lower boundary present a very high oscillating behavior and also, the order of frequency of the two oscillatory boundaries is larger than the order of the height of the domain.

To obtain the homogenized limit problem we proceed in a similar way to the previous case. Indeed, the strong roughness allows us to split the domain in such a way that we can separate the oscillatory boundaries and then we apply the unfolding operator to each part. Note that this case is simpler than the two previous ones. In fact, since the order of the period of the oscillations is smaller than ϵ at the top and bottom boundary we will not need the oscillating functions introduced in the previous cases.

In view of the variational formulation (3.2.2) and since the source term is uniformly bounded, $|||f^\epsilon|||_{L^2(R^\epsilon)} \leq C$, we easily obtain the a priori estimate

$$|||w^\epsilon|||_{H^1(R^\epsilon)} \leq C, \tag{3.3.2}$$

where the constant C does not depend on ϵ.

We can now state the homogenization result.

Theorem 3.3.1. *Let w^ϵ be the solution of problem (3.0.1). Assume that the non-homogeneous term is uniformly bounded, $|||f^\epsilon|||_{L^2(R^\epsilon)} \leq C$ and that there exists $\hat{f} \in L^2(0,1)$ such that $\hat{f}^\epsilon \overset{\epsilon\to 0}{\rightharpoonup} \hat{f}$, $w-L^2(0,1)$, where $\hat{f}^\epsilon(x_1) \equiv \frac{1}{\epsilon}\int_{-\epsilon h(x_1/\epsilon^\alpha)}^{\epsilon g(x_1/\epsilon^\beta)} f^\epsilon(x_1,x_2)\,dx_2$.*

Then, there exists $u_0 \in H^1(0,1)$ such that $|||w^\epsilon - u_0|||_{L^2(R^\epsilon)} \to 0$ and it is the unique solution of the following Neumann problem:

$$\begin{cases} -\dfrac{g_0+h_0}{\mathcal{M}(g)+\mathcal{M}(h)}u_{0xx} + u_0 = \dfrac{\hat{f}}{\mathcal{M}(g)+\mathcal{M}(h)}, & x \in (0,1), \\ u_0'(0) = u_0'(1) = 0. \end{cases} \tag{3.3.3}$$

where as usual $\mathcal{M}(\cdot)$ denotes the mean value of the function, $\mathcal{M}(g) = \frac{1}{L_1}\int_0^{L_1} g(s)\,ds$ and $\mathcal{M}(h) = \frac{1}{L_2}\int_0^{L_2} h(s)\,ds$.

Proof. Dividing the domain suitably is the key point to obtain the homogenized limit problem.

Therefore, since the upper boundary as well as the lower boundary present a strong oscillatory behavior we consider the following three open subsets of R^ϵ

$$\begin{aligned} R_+^\epsilon &= \Big\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ \epsilon g_0 < x_2 < \epsilon\, g(x_1/\epsilon^\beta)\Big\}, \\ R_0^\epsilon &= \Big\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ -\epsilon h_0 < x_2 < \epsilon g_0\Big\} \\ R_-^\epsilon &= \Big\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ -\epsilon h(x_1/\epsilon^\alpha) < x_2 < -\epsilon h_0\Big\}. \end{aligned}$$

Notice that

$$R^\epsilon = \mathrm{Int}\Big(\overline{R_+^\epsilon} \bigcup \overline{R_0^\epsilon} \bigcup \overline{R_-^\epsilon}\Big).$$

In this situation, we keep the notation of the previous sections

$$w_+^\epsilon := w^\epsilon|_{R_+^\epsilon}, \;\; w_0^\epsilon := w^\epsilon|_{R_0^\epsilon} \;\text{ and }\; w_-^\epsilon := w^\epsilon|_{R_-^\epsilon}.$$

Now, we apply the corresponding unfolding operator to the oscillating open sets and the rescaling operator, see Subsection 1.4.1 in Chapter 1, to the non oscillating thin domain.

Then, from the properties of the unfolding operator and the estimate (3.3.2) it straightforward follows that there exist $u_+, u_- \in L^2(0,1)$, $u_1 \in L^2\big((0,1)\times Y_+^*\big)$ and $u_2 \in L^2\big((0,1)\times Y_-^*\big)$ such that

$$\begin{aligned}
&\mathcal{T}_{\epsilon+}(w_+^\epsilon) \overset{\epsilon\to 0}{\rightharpoonup} u_+ \quad \mathrm{w}-L^2\big((0,1)\times Y_+^*\big),\\
&\mathcal{T}_{\epsilon+}\Big(\frac{\partial w_+^\epsilon}{\partial x_1}\Big) \overset{\epsilon\to 0}{\rightharpoonup} u_1 \quad \mathrm{w}-L^2\big((0,1)\times Y_+^*\big),\\
&\mathcal{T}_{\epsilon-}(w_-^\epsilon) \overset{\epsilon\to 0}{\rightharpoonup} u_- \quad \mathrm{w}-L^2\big((0,1)\times Y_-^*\big),\\
&\mathcal{T}_{\epsilon-}\Big(\frac{\partial w_-^\epsilon}{\partial x_1}\Big) \overset{\epsilon\to 0}{\rightharpoonup} u_2 \quad \mathrm{w}-L^2\big((0,1)\times Y_-^*\big).
\end{aligned}$$

where the basic cells are given by

$$\begin{aligned}
Y_+^* &= \{(y_1,y_2)\in\mathbb{R}^2 \;:\; 0<y_1<L_1,\; g_0<y_2<g(y_1)\},\\
Y_-^* &= \{(z_1,z_2)\in\mathbb{R}^2 \;:\; 0<z_1<L_2,\; -h(z_1)<z_2<-h_0\}.
\end{aligned}$$

Moreover, considering test functions of the form

$$\varphi_+^\epsilon(x_1,x_2) = \epsilon^\beta \tilde\varphi\Big(x_1,\frac{x_2}{\epsilon}\Big)\psi\Big(\Big\{\frac{x_1}{\epsilon^\beta}\Big\}_{L_1}\Big), \quad (x_1,x_2)\in R^\epsilon,$$

where $\varphi \in \mathcal{D}\big((0,1)\times(g_0,g_1)\big)$, $\psi \in C_\#^\infty[0,L_1)$ we have that $u_1 = 0$ by the same argument as in the proof of Theorem 1.4.3. Similarly, taking as test function

$$\varphi_-^\epsilon(x_1,x_2) = \epsilon^\alpha \tilde\varphi\Big(x_1,\frac{x_2}{\epsilon}\Big)\psi\Big(\Big\{\frac{x_1}{\epsilon^\alpha}\Big\}_{L_2}\Big), \quad (x_1,x_2)\in R^\epsilon,$$

where $\varphi \in \mathcal{D}\big((0,1)\times(-h_1,-h_0)\big)$, $\psi\in C_\#^\infty[0,L_2)$ we have that $u_2 = 0$.

On the other hand, taking into account the properties of the rescaling operator, see Proposition 1.4.2, we immediately have that there exists $u_0 \in H^1(0,1)$ such that, up to subsequences,

$$\begin{aligned}
&\Pi_\epsilon(w_0^\epsilon) \overset{\epsilon\to 0}{\rightharpoonup} u_0 \quad \mathrm{w}-H^1(R_0),\\
&\Pi_\epsilon\Big(\frac{\partial w_0^\epsilon}{\partial x_1}\Big) \overset{\epsilon\to 0}{\rightharpoonup} \frac{\partial u_0}{\partial x_1} \quad \mathrm{w}-L^2(R_0).
\end{aligned}$$

Moreover, by a simple computation, see (3.2.10) and (3.2.11), we get

$$\lim_{\epsilon\to 0} |||w^\epsilon - u_0|||_{L^2(R^\epsilon)} = 0.$$

In the same way as in the proof of the previous Theorem we may show that $u_0 = u_- = u_+$.

Finally, taking in (3.2.2) as test function $\varphi \in H^1(0,1)$ and due to the properties of the unfolding and the rescaling operator, the convergences above, and the hypothesis on the function f^ϵ we may pass to the limit to get

$$\frac{1}{L_1}\int_{(0,1)\times Y_+^*} u_0\phi\, dx_1 dy_1 dy_2 + \frac{1}{L_2}\int_{(0,1)\times Y_-^*} u_0\phi\, dx_1 dz_1 dz_2 \\ + \int_0^1\int_{-h_0}^{g_0}\left\{\frac{\partial u}{\partial x_1}\frac{\partial \phi}{\partial x_1} + u\phi\right\} dX_2 dx_1 = \int_{(0,1)} \hat{f}\phi\, dx_1,\ \forall \phi \in H^1(0,1).$$

Equivalently, we have

$$\int_0^1 \{(g_0+h_0)\frac{\partial u}{\partial x_1}\frac{\partial \phi}{\partial x_1} + \big(\mathcal{M}(g)+\mathcal{M}(h)\big)u_0\phi\}\, dx_1 = \int_{(0,1)} \hat{f}\phi\, dx_1,\ \forall \phi \in H^1(0,1),$$

which is the weak formulation (3.3.3). $\square$

Remark 3.3.2. *Notice that, the proof of Theorem 3.3.3 is based on the properties of the unfolding for thin domains with strong oscillations, see Section 1.4. Then, note that although we assume that the functions $g(\cdot)$ and $h(\cdot)$ are C^1 this hypothesis can be relaxed as we have shown in Section 1.4.*

Remark 3.3.3. *Note that the limit problem for the case with resonant and fast oscillations, Section 3.1, may also be obtained using the unfolding operator in a similar way to the cases Weak-Fast and Fast-Fast. Indeed, using the properties of the unfolding operator obtained in Chapter 1 and choosing suitable test functions similar to the test functions defined in Section 3.1 it is also possible to get the homogenized limit problem for the Resonant-Fast case.*

3.4. Thin domains with doubly weak oscillatory boundary

In previous sections we have studied the limit behavior of solutions to (3.0.1) in doubly oscillating thin domains where, at least, one of the boundaries presents an extremely high oscillatory behavior. In this section we analyze the case of weak roughness at the two oscillating boundaries. That is, both top and bottom boundary are given by two periodic functions with order of frequency smaller than the characteristic height of the domain.

Thus, the thin domain is given by

$$R^\epsilon = \Big\{(x,y)\in\mathbb{R}^2 \mid x\in(0,1),\ -\epsilon\, h(x/\epsilon^\alpha) < y < \epsilon\, g(x/\epsilon^\beta)\Big\}, \tag{3.4.1}$$

where $0<\alpha,\beta<1$ and the functions g, h satisfy **(H)**, see Figure 3.7.

Indeed, the thickness of the domain has order ϵ while the period of the oscillations is $L_1\epsilon^\alpha$ at the upper boundary and $L_2\epsilon^\beta$ at the lower boundary with

$$\lim_{\epsilon\to 0}\frac{\epsilon}{\epsilon^\alpha} = \lim_{\epsilon\to 0}\frac{\epsilon}{\epsilon^\beta} = 0.$$

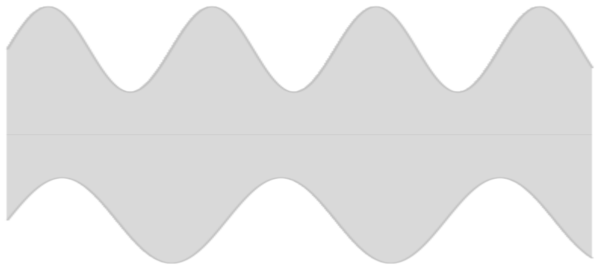


Figure 3.7: Thin domain R^ϵ with doubly weak oscillatory boundary

Note that, in order to simplify the computations and the presentation of the results, in this section we will assume that $f^\epsilon(x_1, x_2) = f(x_1)$ for all $\epsilon > 0$ and $(x_1, x_2) \in R^\epsilon$

As usual, the existence and uniqueness of solution to problem (3.0.1) is guaranteed by Lax-Milgram Theorem for each fixed $\epsilon > 0$ and taking w^ϵ as a test function in the weak formulation of (3.0.1) we easily deduce the a priori estimates

$$|||w^\epsilon|||_{H^1(R^\epsilon)} \le C, \tag{3.4.2}$$

where C is a constant which does not depend on ϵ.

The method used in this section will be different from the one used in previous sections and it is more related to classical works in thin domains with no oscillations, see for instance [73, 100, 7]. The main idea is that since the oscillations at the boundary are not too "wild" we can transform the thin domain R^ϵ into the square $Q = (0,1) \times (0,1)$ with a nice diffeomorphism that will depend on the parameter ϵ. Moreover, the operator Δ will be transformed in another operator in divergence form which will be complicated but since $\alpha, \beta < 1$ we will be able to single out the relevant terms from the terms that will disappear in the limit as $\epsilon \to 0$. This will allow us to reduce the problem to a one dimensional problem and we will be able to pass to the limit in this problem. Passing to the limit in this problem will require some interesting ingredients since for some case we will be dealing with some quasiperiodic functions and oscillating functions with different scales.

As a matter of fact, we have:

Theorem 3.4.1. *Let w^ϵ be the solution of problem (3.0.1) with $f^\epsilon(x_1, x_2) = f(x_1)$. Then, there exists $u_0 \in H^1(0,1)$ such that $|||w^\epsilon - u_0|||_{L^2(R^\epsilon)} \to 0$ and it is the unique solution of the following Neumann problem*

$$\begin{cases} -\dfrac{p_0}{\mathcal{M}(g)+\mathcal{M}(h)} u_{0xx} + u_0 = f, \quad x \in (0,1), \\ u_0'(0) = u_0'(1) = 0, \end{cases} \tag{3.4.3}$$

where the constant p_0 is such that

$$\frac{1}{h\left(\frac{x}{\epsilon^\alpha}\right)+g\left(\frac{x}{\epsilon^\beta}\right)} \overset{\epsilon\to 0}{\rightharpoonup} \frac{1}{p_0} \quad w-L^2(0,1). \tag{3.4.4}$$

Therefore p_0 is given by

$$\frac{1}{p_0}=\begin{cases} \lim\limits_{T\to\infty}\dfrac{1}{T}\displaystyle\int_0^T \frac{1}{g(y)+h(y)}\,dy, & \text{if } \alpha=\beta,\\ \dfrac{1}{L_1L_2}\displaystyle\int_0^{L_1}\int_0^{L_2}\frac{1}{g(y)+h(z)}\,dzdy, & \text{if } \alpha\neq\beta.\end{cases}$$

Remark 3.4.2. *Observe that we distinguish two cases according to $\alpha\neq\beta$ or $\alpha=\beta$.*

In case $\alpha=\beta$ and if L_1 and L_2 are rationally independent the oscillations present a behavior beyond the classical periodic setting. In fact, as we will show in the proof Theorem 3.4.1, the function $\frac{1}{g(y)+h(y)}$ is quasi-periodic. Therefore, in this particular case p_0 is the inverse of the mean value of the function $\frac{1}{g(y)+h(y)}$.

Before proving Theorem 3.4.1, we introduce some important lemmas which will allow us to transform our two-dimensional problem defined in a oscillating thin domain into a one-dimensional problem with oscillating diffusion coefficient. The proof of these lemmas consist essentially in performing suitable change of variables.

First, we show that the study of the limit behavior of the solutions of (3.0.1) is equivalent to analyze the behavior of the solutions of the following problem

$$\begin{cases} -\Delta v^\epsilon+v^\epsilon=f & \text{in } R_a^\epsilon,\\ \dfrac{\partial v^\epsilon}{\partial\nu^\epsilon}=0 & \text{on } \partial R_a^\epsilon,\end{cases} \tag{3.4.5}$$

where ν^ϵ is the unit outward normal to ∂R_a^ϵ and R_a^ϵ is the thin domain with an oscillating boundary, see Figure 3.8, given by

$$R_a^\epsilon=\Big\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1\in(0,1),\ 0<x_2<\epsilon\, g(x_1/\epsilon^\beta)+\epsilon\, h(x_1/\epsilon^\alpha)\Big\}. \tag{3.4.6}$$

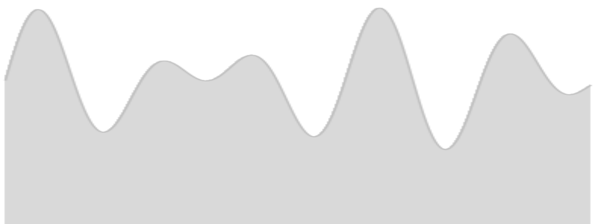


Figure 3.8: Thin domain R_a^ϵ

Observe that the function which describes the oscillating boundary for R_a^ϵ will not necessarily be periodic.

Note that it is easy to prove that the family of solutions of (3.4.5) is also uniformly bounded, that is, there exists a constant C independent of ϵ such that

$$|||v^\epsilon|||_{H^1(R_a^\epsilon)} \leq C. \tag{3.4.7}$$

Lemma 3.4.3. *Let w^ϵ and v^ϵ be the solutions of problems* (3.0.1) *and* (3.4.5) *respectively. Then, considering the following family of diffeomorphisms*

$$\begin{aligned} L^\epsilon : R_a^\epsilon &\longrightarrow R^\epsilon \\ (x_1, x_2) &\longrightarrow (x, y) := (x_1, x_2 - \epsilon\, h(x_1/\epsilon^\alpha)), \end{aligned}$$

we have

$$|||(w^\epsilon \circ L^\epsilon) - v^\epsilon|||_{H^1(R_a^\epsilon)} \stackrel{\epsilon\to 0}{\longrightarrow} 0.$$

Proof. Since in view of definition of L^ϵ we have

$$\begin{aligned} \frac{\partial(w^\epsilon \circ L^\epsilon)}{\partial x_1} &= \frac{\partial w^\epsilon}{\partial x} + \epsilon^{1-\alpha} h'(\frac{x_1}{\epsilon^\alpha})\frac{\partial w^\epsilon}{\partial y}, \\ \frac{\partial(w^\epsilon \circ L^\epsilon)}{\partial x_2} &= \frac{\partial w^\epsilon}{\partial y}, \end{aligned}$$

in the new system of variables ($x_1 = x$ and $x_2 = y + \epsilon\, h(x/\epsilon^\alpha)$) the variational formulation of (3.0.1) is given by

$$\begin{aligned} &\int_{R_a^\epsilon} \Big\{ \frac{\partial(w^\epsilon \circ L^\epsilon)}{\partial x_1}\frac{\partial\varphi}{\partial x_1} + \frac{\partial(w^\epsilon \circ L^\epsilon)}{\partial x_2}\frac{\partial\varphi}{\partial x_2} + (w^\epsilon \circ L^\epsilon)\phi \Big\} dx_1 dx_2 \\ &\quad - \int_{R_a^\epsilon} \epsilon^{1-\alpha} h'(\frac{x_1}{\epsilon^\alpha}) \Big\{ \frac{\partial(w^\epsilon \circ L^\epsilon)}{\partial x_2}\frac{\partial\varphi}{\partial x_1} + \frac{\partial(w^\epsilon \circ L^\epsilon)}{\partial x_1}\frac{\partial\varphi}{\partial x_2} \Big\} dx_1 dx_2 \\ &\quad + \int_{R_a^\epsilon} \Big(\epsilon^{1-\alpha} h'(\frac{x_1}{\epsilon^\alpha})\Big)^2 \frac{\partial(w^\epsilon \circ L^\epsilon)}{\partial x_2}\frac{\partial\varphi}{\partial x_2} \Big] dx_1 dx_2 \\ &= \int_{R_a^\epsilon} f\varphi\, dx_1 dx_2, \quad \forall \varphi \in H^1(R_a^\epsilon). \end{aligned} \tag{3.4.8}$$

On the other hand, the weak formulation of (3.4.5) is: find $v^\epsilon \in H^1(R_a^\epsilon)$ such that

$$\int_{R_a^\epsilon} \Big\{ \frac{\partial v^\epsilon}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{\partial v^\epsilon}{\partial x_2}\frac{\partial \varphi}{\partial x_2} + v^\epsilon\varphi \Big\}\, dx_1 dx_2 = \int_{R_a^\epsilon} f\varphi\, dx_1 dx_2, \quad \forall\varphi \in H^1(R_a^\epsilon). \tag{3.4.9}$$

Therefore, subtracting (3.4.9) from (3.4.8), taking $(w^\epsilon \circ L^\epsilon) - v^\epsilon$ as a test function and by the uniform bounds (3.4.2) and (3.4.7) we obtain

$$\begin{aligned} &\int_{R_a^\epsilon} \Big\{ \Big(\frac{\partial(w^\epsilon \circ L^\epsilon)}{\partial x_1} - \frac{\partial v^\epsilon}{\partial x_1} \Big)^2 + \Big(\frac{\partial(w^\epsilon \circ L^\epsilon)}{\partial x_2} - \frac{\partial v^\epsilon}{\partial x_2} \Big)^2 \Big\}\, dx_1 dx_2 \\ &\qquad + \int_{R_a^\epsilon} \big((w^\epsilon \circ L^\epsilon) - v^\epsilon \big)^2\, dx_1 dx_2 \leq C\epsilon^{1-\alpha}. \end{aligned}$$

Hence, since $0 < \alpha < 1$ the result is proved. □

Remark 3.4.4. *Notice that from the point of view of the limit behavior of the solutions it is the same to study the problem* (3.0.1) *defined in a doubly oscillating thin domain as the problem* (3.4.5) *with the same differential operator but defined in a thin domain with just one oscillating boundary.*

Now we define a transformation on the thin domain R_1^ϵ, which will map R_1^ϵ into the fixed rectangle $Q = (0,1) \times (0,1)$. This transformation is given by

$$\begin{aligned} F^\epsilon : Q &\longrightarrow R_a^\epsilon \\ (x,y) &\longrightarrow (x_1, x_2) := (x, y\, \epsilon G_\epsilon(x)), \end{aligned}$$

where $G_\epsilon(x) = g(x/\epsilon^\beta) + h(x/\epsilon^\alpha)$. Notice that,

$$0 < g_0 + h_0 \leq G_\epsilon(x) \leq g_1 + h_1, \quad \text{ for all } x \in (0,1) \tag{3.4.10}$$

Then, under the change of variables

$$x_1 = x, \quad x_2 = y\, \epsilon G_\epsilon(x),$$

the problem (3.4.9) becomes

$$\begin{cases} -\dfrac{1}{G_\epsilon} \mathrm{div}\big(B^\epsilon(u^\epsilon)\big) + u^\epsilon = f \quad \text{ in } Q, \\ \qquad B(u^\epsilon)\cdot \eta = 0 \quad \text{ on } \partial Q, \\ \qquad u^\epsilon = v^\epsilon \circ F^\epsilon \text{ in } Q, \end{cases} \tag{3.4.11}$$

where η denotes the unit outward normal vector field to ∂Q and

$$B(u^\epsilon) = \Big(G_\epsilon \frac{\partial u^\epsilon}{\partial x} - yG'_\epsilon \frac{\partial u^\epsilon}{\partial y}, -yG'_\epsilon \frac{\partial u^\epsilon}{\partial x} + \Big(\frac{(yG'_\epsilon)^2}{G_\epsilon} + \frac{1}{\epsilon^2 G_\epsilon} \Big) \frac{\partial u^\epsilon}{\partial y} \Big).$$

see for instance [73, 100].

Notice that in the new system of coordinates we obtain a domain which is neither thin nor oscillating anymore. In some sense, we have substituted the oscillating thin domain by non constant coefficients in the differential operator.

Taking into account that v^ϵ satisfies (3.4.7) and the assumptions on the periodic functions g and h we get the following estimates

$$||u^\epsilon||^2_{L^2(Q)} = \int_Q |u^\epsilon|^2 \, dxdy = \int_Q |v^\epsilon \circ F^\epsilon|^2 \, dxdy = \int_{R_a^\epsilon} \frac{|v^\epsilon|^2}{\epsilon G_\epsilon} \leq C|||v^\epsilon|||^2_{L^2(R_a^\epsilon)} \leq C, \tag{3.4.12}$$

$$\begin{aligned} \Big|\Big| \frac{\partial u^\epsilon}{\partial x} \Big|\Big|^2_{L^2(Q)} &= \int_Q \Big| \frac{\partial (v^\epsilon \circ F^\epsilon)}{\partial x} \Big|^2 dxdy = \int_{R_a^\epsilon} \frac{1}{\epsilon G_\epsilon} \Big| \frac{\partial v^\epsilon}{\partial x_1} + \frac{\partial v^\epsilon}{\partial x_2} \frac{G'_\epsilon x_2}{G_\epsilon} \Big|^2 dx_1 dx_2 \\ &\leq C\Big(\Big|\Big|\Big| \frac{\partial v^\epsilon}{\partial x_1} \Big|\Big|\Big|^2_{L^2(R_a^\epsilon)} + \Big|\Big|\Big| \frac{\partial v^\epsilon}{\partial x_2} \Big|\Big|\Big|^2_{L^2(R_a^\epsilon)} \Big) \leq C, \end{aligned} \tag{3.4.13}$$

$$\Big|\Big| \frac{\partial u^\epsilon}{\partial y} \Big|\Big|^2_{L^2(Q)} = \int_Q \Big| \frac{\partial (v^\epsilon \circ F^\epsilon)}{\partial y} \Big|^2 dxdy = \int_{R_a^\epsilon} \frac{1}{\epsilon G_\epsilon} \Big| \frac{\partial v^\epsilon}{\partial x_2} \epsilon G_\epsilon \Big|^2 dx_1 dx_2$$

$$\leq \epsilon^2 C \Big|\Big|\Big| \frac{\partial v^\epsilon}{\partial x_2} \Big|\Big|\Big|^2_{L^2(R_a^\epsilon)} \leq \epsilon^2 C. \tag{3.4.14}$$

In order to analyze the limit behavior of the solutions of (3.4.11) we establish the relation to the solutions of and the following problem

$$\begin{cases} -\dfrac{1}{G_\epsilon}\Big(\dfrac{\partial}{\partial x}\Big(G_\epsilon \dfrac{\partial w_1^\epsilon}{\partial x}\Big) + \dfrac{1}{\epsilon^2 G_\epsilon}\dfrac{\partial^2 w_1^\epsilon}{\partial y^2} + w_1^\epsilon = f \quad \text{in } Q, \\ \qquad\qquad\qquad\qquad \dfrac{\partial w_1^\epsilon}{\partial \eta} = 0 \quad \text{on } \partial Q. \end{cases} \tag{3.4.15}$$

Observe that under the assumptions on the functions g and h, equation (3.4.15) admits a unique solution $w_1^\epsilon \in H^1(Q)$, which satisfies the a priori estimates

$$\|w_1^\epsilon\|_{L^2(Q)}, \ \Big\|\frac{\partial w_1^\epsilon}{\partial x}\Big\|_{L^2(Q)}, \ \frac{1}{\epsilon}\Big\|\frac{\partial w_1^\epsilon}{\partial y}\Big\|_{L^2(Q)} \leq C. \tag{3.4.16}$$

Lemma 3.4.5. *Let u^ϵ and w_1^ϵ be the solution of problems* (3.4.11) *and* (3.4.15) *respectively. Then, we have*

$$\Big|\Big|\frac{\partial(u^\epsilon - w_1^\epsilon)}{\partial x}\Big|\Big|_{L^2(Q)} + \frac{1}{\epsilon^2}\Big|\Big|\frac{\partial(u^\epsilon - w_1^\epsilon)}{\partial y}\Big|\Big|_{L^2(Q)} + ||u^\epsilon - w_1^\epsilon||_{L^2(Q)} \stackrel{\epsilon\to 0}{\longrightarrow} 0.$$

Proof. Subtracting the weak formulation of (3.4.15) from the weak formulation of (3.4.11) and choosing $u^\epsilon - w_1^\epsilon$ as test function we get

$$\begin{aligned} &\int_Q \Big\{ G_\epsilon \Big(\frac{\partial(u^\epsilon - w_1^\epsilon)}{\partial x}\Big)^2 + \frac{1}{\epsilon^2 G_\epsilon}\Big(\frac{\partial(u^\epsilon - w_1^\epsilon)}{\partial y}\Big)^2 + \big(u^\epsilon - w_1^\epsilon\big)^2\Big\} dxdy \\ &\quad = \int_Q G'_\epsilon y \frac{\partial u^\epsilon}{\partial y}\frac{\partial(u^\epsilon - w_1^\epsilon)}{\partial x} + G'_\epsilon y \frac{\partial u^\epsilon}{\partial x}\frac{\partial(u^\epsilon - w_1^\epsilon)}{\partial y} - \frac{(G'_\epsilon y)^2}{G_\epsilon}\frac{\partial u^\epsilon}{\partial y}\frac{\partial(u^\epsilon - w_1^\epsilon)}{\partial y} dxdy. \end{aligned} \tag{3.4.17}$$

Taking into account that $G'_\epsilon = \frac{1}{\epsilon^\beta} g\big(\frac{x}{\epsilon^\beta}\big) + \frac{1}{\epsilon^\alpha} h\big(\frac{x}{\epsilon^\alpha}\big)$ and the a priori estimates of u^ϵ and w_1^ϵ, see (3.4.12), (3.4.13), (3.4.14) and (3.4.16), we can ensure that the right-hand side satisfies

$$\begin{aligned} &\Big| \int_Q G'_\epsilon y \frac{\partial u^\epsilon}{\partial y}\frac{\partial(u^\epsilon - w_1^\epsilon)}{\partial x} + G'_\epsilon y \frac{\partial u^\epsilon}{\partial x}\frac{\partial(u^\epsilon - w_1^\epsilon)}{\partial y} - \frac{(G'_\epsilon y)^2}{G_\epsilon}\frac{\partial u^\epsilon}{\partial y}\frac{\partial(u^\epsilon - w_1^\epsilon)}{\partial y} dxdy \Big| \\ &\leq C \max\{\epsilon^{1-\alpha}, \epsilon^{1-\beta}\} \frac{1}{\epsilon^2}\Big|\Big|\frac{\partial u^\epsilon}{\partial y}\Big|\Big|^2_{L^2(Q)} \Big|\Big|\frac{\partial(u^\epsilon - w_1^\epsilon)}{\partial x}\Big|\Big|^2_{L^2(Q)} \\ &\quad + C \max\{\epsilon^{1-\alpha}, \epsilon^{1-\beta}\} \frac{1}{\epsilon^2}\Big|\Big|\frac{\partial(u^\epsilon - w_1^\epsilon)}{\partial y}\Big|\Big|^2_{L^2(Q)} \Big|\Big|\frac{\partial(u^\epsilon)}{\partial x}\Big|\Big|^2_{L^2(Q)} \\ &\quad + C \max\{\epsilon^{2-2\alpha}, \epsilon^{2-2\beta}\} \frac{1}{\epsilon^2}\Big|\Big|\frac{\partial u^\epsilon}{\partial y}\Big|\Big|^2_{L^2(Q)} \frac{1}{\epsilon^2}\Big|\Big|\frac{\partial(u^\epsilon - w_1^\epsilon)}{\partial y}\Big|\Big|^2_{L^2(Q)} \\ &\leq C \max\{\epsilon^{1-\alpha}, \epsilon^{1-\beta}\}. \end{aligned}$$

Therefore, from (3.4.17) and (3.4.18) we obtain the result

$$\int_Q \Big\{ G_\epsilon \Big(\frac{\partial(u^\epsilon - w_1^\epsilon)}{\partial x}\Big)^2 + \frac{1}{\epsilon^2 G_\epsilon}\Big(\frac{\partial(u^\epsilon - w_1^\epsilon)}{\partial y}\Big)^2 + G_\epsilon\big(u^\epsilon - w_1^\epsilon\big)^2\Big\} dxdy \stackrel{\epsilon\to 0}{\longrightarrow} 0.$$

This last expression together with (3.4.10) show the result. $\square$

Notice that problem (3.4.15) is of separate variables. Moreover, since the domain is a rectangle, we will be able to separate variables and since f depends only on the variable x, we will have that necessarily w_1^ϵ is the solution of the following problem

$$\begin{cases} -\dfrac{1}{G_\epsilon}\Big(\dfrac{\partial}{\partial x}\Big(G_\epsilon \dfrac{\partial u_1^\epsilon}{\partial x}\Big)\Big) + u_1^\epsilon = f & \text{in } (0,1), \\ {u_1^\epsilon}'(0) = {u_1^\epsilon}'(1) = 0. \end{cases} \tag{3.4.18}$$

To see this, just notice that (3.4.18) has a unique weak solution which obviously will depend only on the variable x. But by direct computation this function will also be a solution of problem (3.4.15).

Now we are in conditions to prove the main result, Theorem 3.4.1.

Proof. First of all, notice that Lemma 3.4.3 and Lemma 3.4.5 allows us to reduce the proof to the study of the asymptotic behavior of the solutions of the one dimensional case (3.4.18). Then, it is enough to obtain the homogenized limit equation for the simpler problem (3.4.18). However, we would like to point that (3.4.18) presents the particularity of having not necessarily periodic coefficients.

The weak formulation of (3.4.18) is given by

$$\int_0^1 \Big\{G_\epsilon \frac{\partial u_1^\epsilon}{\partial x}\frac{\partial \phi^\epsilon}{\partial x} + G_\epsilon u_1^\epsilon \phi\Big\}\, dx = \int_0^1 G_\epsilon f \phi, \quad \text{for all } \phi \in H^1(0,1) \tag{3.4.19}$$

We start by establishing a priori estimates of u_1^ϵ. Considering u_1^ϵ as a test function in (3.4.19), we get

$$\int_0^1 \Big\{G_\epsilon \Big(\frac{\partial u_1^\epsilon}{\partial x}\Big)^2 + G_\epsilon {u_1^\epsilon}^2\Big\} dx \le ||f||_{L^2(0,1)} ||u_1^\epsilon||_{L^2(0,1)}.$$

Then, taking into account (3.4.10) and $||f||_{L^2(0,1)} \le C$ we deduce

$$||u_1^\epsilon||_{H^1(0,1)} \le C.$$

Thus, by weak compactness there exists $u_0 \in H^1(0,1)$ such that, up to subsequences

$$u_1^\epsilon \overset{\epsilon\to 0}{\rightharpoonup} u_0 \quad w - H^1(0,1). \tag{3.4.20}$$

As in the simplest cases for the homogenization, see for example [19, 44], the key question now is: How is the limit of the product $G_\epsilon \frac{\partial u_1^\epsilon}{\partial x}$?

To solve this, we first obtain the weak limit of the functions G_ϵ and $\dfrac{1}{G_\epsilon}$.

On one hand, since $G_\epsilon(x) = g(x/\epsilon^\beta) + h(x/\epsilon^\alpha)$ is the sum of two periodic functions it is obvious from the Average Convergence for Periodic Functions Theorem (see, e.g., [52, p. xvi]) that $G_\epsilon(x)$ converges in a weak sense to the sum of the corresponding mean values, that is,

$$G_\epsilon \overset{\epsilon\to 0}{\rightharpoonup} \frac{1}{L_1}\int_0^{L_1} g(y)\, dy + \frac{1}{L_2}\int_0^{L_2} h(z)\, dz \equiv \mathcal{M}(g) + \mathcal{M}(h) \quad w - L^2(0,1). \tag{3.4.21}$$

On the other hand, let us assume that we have shown that

$$\frac{1}{G_\epsilon} = \frac{1}{g(\frac{\cdot}{\epsilon^\beta}) + h(\frac{\cdot}{\epsilon^\beta})} \overset{\epsilon \to 0}{\rightharpoonup} \frac{1}{p_0}$$

where p_0 is defined as in the statement of Theorem 3.4.1.

Now we use a classical argument to get the convergence of the product $G_\epsilon \frac{\partial u_1^\epsilon}{\partial x}$. Observe that $G_\epsilon \frac{\partial u_1^\epsilon}{\partial x}$ is uniformly bounded in $L^2(0,1)$ since

$$\Big|\Big|\frac{\partial u_1^\epsilon}{\partial x}\Big|\Big|_{L^2(0,1)} \leq C \quad \text{and} \quad 0 < G_\epsilon(x) < g_1 + h_1, \text{ for each } x \in (0,1).$$

Moreover, taking into account that

$$\frac{\partial}{\partial x}\Big(G_\epsilon \frac{\partial u_1^\epsilon}{\partial x}\Big) = -fG_\epsilon + G_\epsilon u_1^\epsilon,$$

we deduce that $G_\epsilon \frac{\partial u_1^\epsilon}{\partial x}$ is uniformly bounded in $H^1(0,1)$. Then, it follows that there exists a function σ such that, up to subsequences,

$$G_\epsilon \frac{\partial u_1^\epsilon}{\partial x} \longrightarrow \sigma \quad \text{strongly in } L^2(0,1).$$

Thus,

$$\frac{\partial u_1^\epsilon}{\partial x} = \frac{1}{G_\epsilon}\Big(G_\epsilon \frac{\partial u_1^\epsilon}{\partial x}\Big) \overset{\epsilon \to 0}{\rightharpoonup} \frac{1}{p_0}\sigma \quad w - L^2(0,1).$$

Consequently, due to convergence (3.4.20) we have

$$\frac{\partial u_0}{\partial x} = \frac{1}{p_0}\sigma,$$

or equivalently,

$$G_\epsilon \frac{\partial u_1^\epsilon}{\partial x} \overset{\epsilon \to 0}{\longrightarrow} p_0 \frac{\partial u_0}{\partial x} \quad \text{strongly in } L^2(0,1). \tag{3.4.22}$$

Therefore, in view of (3.4.20), and (3.4.22) we can pass to the limit in (3.4.19)

$$\int_0^1 \Big\{p_0 \frac{\partial u_0}{\partial x}\frac{\partial \phi}{\partial x} + (\mathcal{M}(g) + \mathcal{M}(h))u_0\phi\} \, dx = \int_0^1 (\mathcal{M}(g) + \mathcal{M}(h)) f\phi \, dx,$$

which is the weak formulation of (3.4.3).

To conclude the proof of the theorem we need to calculate the weak limit of $\frac{1}{G_\epsilon}$. We distinguish two cases:

i) Same order of oscillation ($\alpha = \beta$).

$$\frac{1}{g\Big(\frac{x}{\epsilon^\alpha}\Big) + h\Big(\frac{x}{\epsilon^\alpha}\Big)} \overset{\epsilon \to 0}{\rightharpoonup} \frac{1}{p_0} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{g(y) + h(y)} \, dy \quad w - L^2(0,1). \tag{3.4.23}$$

We treat here the case where the function G_ϵ presents only one small scale, that is,

$$\frac{1}{G_\epsilon(x)} = \frac{1}{g(x/\epsilon^\alpha) + h(x/\epsilon^\alpha)}, \quad \text{for } x \in (0,1) \text{ and } \alpha \in (0,1).$$

Note that if the periods L_1 and L_2 are rationally dependent, there exist $p, q \in \mathbb{N}$ such that $pL_1 = qL_2$, then we immediately have from the Average Convergence for Periodic Functions (see, e.g., [52, p. xvi]) the weak convergence of $\frac{1}{G_\epsilon}$

$$\frac{1}{G_\epsilon} \overset{\epsilon\to 0}{\rightharpoonup} \frac{1}{pL_1} \int_0^{pL_1} \frac{1}{g(y) + h(y)}\, dy \quad w - L^2(0,1).$$

However, if L_1 and L_2 are rationally independent the usual periodicity hypothesis is replaced by a more general behavior: almost periodicity, see for example [21, 27]. Indeed, in this case the function $\frac{1}{G(y)} = \frac{1}{g(y)+h(y)}$ is not periodic since there exists no value L which satisfies

$$\frac{1}{G(y+L)} = \frac{1}{G(y)} \quad \forall y \in \mathbb{R},$$

but we then show that it is almost periodic which allows us to obtain the weak limit.

Since $G(y) = g(y)+h(y)$ is the sum of two periodic functions with different period we can ensure that G is an almost periodic function. Then, from the definition of almost periodicity, for every $\epsilon > 0$ there exists $T_0(\epsilon)$ such that every interval of length $T_0(\epsilon)$ contains a number τ with the following property:

$$|G(y+\tau) - G(y)| \leq m^2\epsilon, \quad \text{for each } y \in \mathbb{R},$$

where m is a constant such that $0 < m \leq g(y) + h(y)$, $\forall y \in \mathbb{R}$.

So we have,

$$\Big|\frac{1}{G(y+\tau)} - \frac{1}{G(y)}\Big| = \Big|\frac{G(y) - G(y+\tau)}{G(y+\tau)G(y)}\Big| \leq \frac{m^2\epsilon}{m^2} = \epsilon,$$

and hence $\frac{1}{G(y)}$ is almost periodic.

Therefore, note that $\displaystyle\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{g(y)+h(y)}\, dy$ is well defined since it is the mean value of the almost periodic periodic function $\frac{1}{G(y)}$.

Now, we are in conditions to prove the desired weak convergence (3.4.23).

To obtain (3.4.23), since $||\frac{1}{G(y)}||_{L^\infty(0,1)} \leq \frac{1}{g_0+h_0}$ and the set of all the step functions is dense in $L^p(0,1), 1 \leq p < \infty$, it is enough to prove

$$\lim_{\epsilon\to 0} \int_a^b \frac{1}{g\Big(\frac{x}{\epsilon^\alpha}\Big) + h\Big(\frac{x}{\epsilon^\alpha}\Big)} \overset{\epsilon\to 0}{\longrightarrow} (b-a) \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{g(y)+h(y)}\, dy, \quad \text{for any } (a,b) \subset (0,1). \tag{3.4.24}$$

We can write

$$\int_a^b \frac{1}{g\Big(\frac{x}{\epsilon^\alpha}\Big) + h\Big(\frac{x}{\epsilon^\alpha}\Big)}\, dx = \int_0^b \frac{1}{g\Big(\frac{x}{\epsilon^\alpha}\Big) + h\Big(\frac{x}{\epsilon^\alpha}\Big)}\, dx - \int_0^a \frac{1}{g\Big(\frac{x}{\epsilon^\alpha}\Big) + h\Big(\frac{x}{\epsilon^\alpha}\Big)}\, dx. \tag{3.4.25}$$

By a simple change of variables we have

$$\int_0^e \frac{1}{g\Big(\frac{x}{\epsilon^\alpha}\Big)+h\Big(\frac{x}{\epsilon^\alpha}\Big)}\,dx = e\frac{\epsilon^\alpha}{e}\int_0^{e/\epsilon^\alpha}\frac{1}{g(y)+h(y)}\,dy, \quad \forall e\in(0,1).$$

Then, since $\dfrac{1}{g(y)+h(y)}\,dy$ is almost periodic we can pass to the limit at the right-hand side of the last equality above to get

$$\lim_{\epsilon\to 0}\int_0^e \frac{1}{g\Big(\frac{x}{\epsilon^\alpha}\Big)+h\Big(\frac{x}{\epsilon^\alpha}\Big)}\,dx \overset{\epsilon\to 0}{\longrightarrow} e\lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{1}{g(y)+h(y)}\,dy, \quad \forall e\in(0,1). \tag{3.4.26}$$

Finally, from (3.4.25) and (3.4.26) we get convergence (3.4.24).

ii) Different order of oscillation ($\alpha\neq\beta$).

$$\frac{1}{g\Big(\frac{x}{\epsilon^\beta}\Big)+h\Big(\frac{x}{\epsilon^\alpha}\Big)} \overset{\epsilon\to 0}{\rightharpoonup} \frac{1}{p_0} = \frac{1}{L_1L_2}\int_0^{L_1}\int_0^{L_2}\frac{1}{g(y)+h(z)}\,dzdy\ w-L^2(0,1). \tag{3.4.27}$$

Observe that in this case we are dealing with two microscopic scales which is a generalization of the classical result for periodic functions. Although the result is known in the literature, see e.g. [19], we are going to give a proof using the notation introduced in Chapter 1 for the unfolding operator.

Note that using the same arguments as the previous paragraph it is enough to prove that

$$\lim_{\epsilon\to 0}\int_0^b \frac{1}{g\Big(\frac{x}{\epsilon^\alpha}\Big)+h\Big(\frac{x}{\epsilon^\alpha}\Big)}\,dx \overset{\epsilon\to 0}{\longrightarrow} \frac{b}{L_1L_2}\int_0^{L_1}\int_0^{L_2}\frac{1}{g(y)+h(z)}\,dzdy, \quad \forall b\in(0,1).$$

To prove this, we perform first the change of scale corresponding to the oscillations of order $\epsilon^\beta L_1$ and then, we perform the unfolding change of scale associated to the oscillations of order $\epsilon^\alpha L_2$ with a microscopic correction. See [86, 87] for more general results in reiterated homogenization using the unfolding periodic method.

First of all, we recall some properties of the unfolding change of scale which we have proved in Chapter 1. Observe that from a simple change of variables we obtain

$$\frac{1}{L_1}\int_0^b\int_0^L \phi(\delta\big[\frac{x}{\delta}\big]_L L+\delta y_1)dy_1dy_2 - \int_b^{\delta L(N_\delta+2)}\phi dx = \int_0^b \phi\,dx,\ \forall\phi\in C^1(\mathbb{R}), \tag{3.4.28}$$

where, using the same notation as Chapter 1, L is any strictly positive number, δ is sufficiently close to zero, N_δ is the largest integer such that $(N_\delta+1)L\delta\leq b$ and $[x]_L$ denotes the unique integer such that $x\in\big[[x]_L L,([x]_L+1)L\big)$ and $\{x\}_L$ is such that $x=[x]_L L+\{x\}_L$. Moreover since $\phi\in C^1(\mathbb{R})$ it follows that

$$\lim_{\delta\to 0}\frac{1}{L}\int_0^b\int_0^L \phi(\delta\big[\frac{x}{\delta}\big]_L L+\delta y_1)dy_1dy_2 = \int_0^b \phi\,dx,\ \forall\phi\in C^1(\mathbb{R}).$$

Then, considering $L = L_1$, $\delta = \epsilon^\alpha$ and $\phi = \frac{1}{g\left(\frac{x}{\epsilon^\beta}\right)+h\left(\frac{x}{\epsilon^\alpha}\right)}$ we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^b \frac{dx}{g\left(\frac{x}{\epsilon^\beta}\right)+h\left(\frac{x}{\epsilon^\alpha}\right)} \\
&\qquad = \lim_{\epsilon\to 0}\int_0^b \frac{1}{L_1}\int_0^{L_1} \frac{dy_1 dx}{g\left(\left[\frac{x}{\epsilon^\beta}\right]_{L_1} L_1 + y_1\right) + h\left(\frac{1}{\epsilon^{\alpha-\beta}}\left[\frac{x}{\epsilon^\beta}\right]_{L_1} L_1 + \frac{y_1}{\epsilon^{\alpha-\beta}}\right)} \\
&\qquad = \lim_{\epsilon\to 0}\int_0^b \frac{1}{L_1}\int_0^{L_1} \frac{dy_1 dx}{g(y_1) + h\left(\frac{1}{\epsilon^{\alpha-\beta}}\left[\frac{x}{\epsilon^\beta}\right]_{L_1} L_1 + \frac{y_1}{\epsilon^{\alpha-\beta}}\right)}.
\end{aligned} \tag{3.4.29}$$

Now, by a simple change of variables, $z_1 = y_1 + \delta_\epsilon(x)$ with $\delta_\epsilon(x) = \epsilon^{\alpha-\beta}\left\{\frac{1}{\epsilon^{\alpha-\beta}}\left[\frac{x}{\epsilon^\alpha}\right]_{L_1} L_1\right\}_{L_2}$, and taking into account that

$$\{z\}_{L_2} = z - [z]_{L_2} L_2 \quad \forall z \in \mathbb{R},$$

we may rewrite the last term of the equality (3.4.29) as follows

$$\begin{aligned}
&\int_0^b \frac{1}{L_1}\int_0^{L_1} \frac{dy_1 dx}{g(y_1) + h\left(\frac{1}{\epsilon^{\alpha-\beta}}\left[\frac{x}{\epsilon^\alpha}\right]_{L_1} L_1 + \frac{y_1}{\epsilon^{\alpha-\beta}}\right)} \\
&= \int_0^b \frac{1}{L_1}\int_{\delta_\epsilon(x)}^{L_1+\delta_\epsilon(x)} \frac{dz_1 dx}{g(z_1 - \delta_\epsilon(x)) + h\left(\frac{z_1}{\epsilon^{\alpha-\beta}}\right)}.
\end{aligned} \tag{3.4.30}$$

Then, from (3.4.29) and (3.4.30) we have

$$\lim_{\epsilon\to 0}\int_0^b \frac{1}{g\left(\frac{x}{\epsilon^\beta}\right)+h\left(\frac{x}{\epsilon^\alpha}\right)} = \lim_{\epsilon\to 0}\int_0^b \frac{1}{L_1}\int_{\delta_\epsilon(x)}^{L_1+\delta_\epsilon(x)} \frac{dz_1 dx}{g(z_1 - \delta_\epsilon(x)) + h\left(\frac{z_1}{\epsilon^{\alpha-\beta}}\right)}.$$

Now using property (3.4.28) for the variable z_1, considering $L = L_2$ and $\delta = \epsilon^{\alpha-\beta}$ we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^b \frac{1}{g\left(\frac{x}{\epsilon^\beta}\right)+h\left(\frac{x}{\epsilon^\alpha}\right)} \\
&\quad = \lim_{\epsilon\to 0}\int_0^b \frac{1}{L_1}\int_{\delta_\epsilon(x)}^{L_1+\delta_\epsilon(x)} \frac{1}{L_2}\int_0^{L_2} \frac{dz_2 dz_1 dx}{g\left(\epsilon^{\alpha-\beta}\left[\frac{z_1}{\epsilon^{\alpha-\beta}}\right]_{L_2} L_2 + \epsilon^{\alpha-\beta} z_2 - \delta_\epsilon(x)\right) + h(z_2)}.
\end{aligned}$$

Finally, taking into account the following convergences

$$\delta_\epsilon(x) \xrightarrow{\epsilon\to 0} 0, \text{ a.e } x \in (0,1),$$

$$\epsilon^{\alpha-\beta}\left[\frac{z_1}{\epsilon^{\alpha-\beta}}\right]_{L_2} L_2 + \epsilon^{\alpha-\beta} z_2 \xrightarrow{\epsilon\to 0} z_1, \text{ a.e } (z_1, z_2) \in (0, L_1) \times (0, L_2),$$

which follows easily from the definition of each term, the continuity of the function g and h and with the aid of the Lebesgue's Dominated Convergence Theorem, we may pass to the limit at the right-hand side of the last equality to get

$$\lim_{\epsilon\to 0}\int_0^b \frac{1}{g\left(\frac{x}{\epsilon^\alpha}\right)+h\left(\frac{x}{\epsilon^\beta}\right)} = \lim_{\epsilon\to 0}\int_0^b \frac{1}{L_1L_2}\int_0^{L_1}\int_0^{L_2}\frac{dz_2dz_1dx}{g(z_1)+h(z_2)},$$

which proves (3.4.27). This concludes the proof of the theorem. □

Remark 3.4.6. *For simplicity we have stated the result for two periodic functions, g and h, but we could consider more general situations without any essential change in the proof of the homogenization result.*

Thus, we may include the case where the amplitude of the oscillations vary with respect to $x \in (0,1)$. If we consider $G_\epsilon(x) = g(x, x/\epsilon^\alpha) + h(x, x/\epsilon^\beta)$ for $\alpha, \beta \in (0,1)$ and we assume that

$$G_\epsilon(\cdot) \overset{\epsilon\to 0}{\rightharpoonup} m(\cdot) \quad w - L^2(0,1) \;\; \text{and} \;\; \frac{1}{G_\epsilon(\cdot)} \overset{\epsilon\to 0}{\rightharpoonup} \frac{1}{p(\cdot)} \quad w - L^2(0,1),$$

then the limit problem is

$$\begin{cases} -\dfrac{1}{m(x)}\Big(p(x)u_{0x}\Big)_x + u_0 = f, \quad x \in (0,1), \\ \qquad\qquad u_0'(0) = u_0'(1) = 0. \end{cases}$$

For instance, we may think that the amplitude of the oscillations is modulated by a function. If we consider $G_\epsilon(x) = a(x) + g(x/\epsilon^\alpha) + h(x/\epsilon^\beta)$ we have

$$G_\epsilon(\cdot) \overset{\epsilon\to 0}{\rightharpoonup} m(\cdot) = a(\cdot) + \frac{1}{L_1}\int_0^{L_1} g(y)\,dy + \frac{1}{L_2}\int_0^{L_2} h(z)\,dz, \quad \text{for any } \alpha, \beta \in (0,1),$$

$$\frac{1}{G_\epsilon(\cdot)} \overset{\epsilon\to 0}{\rightharpoonup} \frac{1}{p(\cdot)} = \begin{cases} \displaystyle\lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{1}{a(\cdot)+g(y)+h(y)}\,dy, & \text{if } \alpha = \beta, \\ \displaystyle\frac{1}{L_1L_2}\int_0^{L_1}\int_0^{L_2}\frac{1}{a(\cdot)+g(y)+h(z)}\,dzdy, & \text{if } \alpha \neq \beta. \end{cases}$$

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